SUBDIVISION IN 1D.

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1. Facts

A Stationary subdivision scheme S with subdivision mask σ_k is defined via:

$$f_{j+1,l} = \sum_k \sigma_{l-2k} f_{j,k}.$$

B Linear spline $(\ell_j g)(t) = \sum_k g_k B_1(2^j t - k + 1)$.

C Multi-level sequence $(f_{j,k})_{j \geq 0}$ produces $F(t)$ if $F(t) = \lim_{j\to\infty}(\ell_j f_j)(t)$. When $(f_{j,k})$ comes from a subdivision scheme S we write $S^\infty[f_0] = F$.

D The scaling function, or the fundamental solution of the subdivision scheme is $\varphi(t) = S^\infty[\delta^0](t)$.

E Refinement equation $\phi(t) = \sum_l \sigma_l \phi(2t - l)$.

F Let ϕ and ξ be the fundamental solutions of subdivision schemes with masks σ^ϕ and σ^ξ correspondingly. Then $(\phi \star \xi)(t) = \int \phi(t-s)\xi(s)ds$ is the fundamental solution of the subdivision scheme with the mask $\sigma^\phi \star \sigma^\xi/2$ where $(\sigma^\phi \star \sigma^\xi)_q = \sum_k \sigma^\phi_{q-k}\sigma^\xi_k$.

G z-transform: $f(z) = \sum_k f_k z^k$.

H B_0 has mask $1 + z$.

I If $f \in C^{n-1}$ then $f \star B_0 \in C^n$.

J Subdivision scheme in z:

$$f_{j+1}(z) = \sigma(z) f_j(z^2).$$

K Affine invariance $S\mathbf{1} = \mathbf{1}$. Hence $\sum_k \sigma_{2k} = \sum_k \sigma_{2k+1} = 1$, and therefore $\sigma(-1) = 0$ and then -1 is a root of polynomial $\sigma(z)$ so that $\sigma(z) = (1+z)\tau(z)$.

L Differences $(\Delta g)_k := g_k - g_{k-1}$, its z-transform $(\Delta g)(z) = (1-z)g(z)$. Derived subdivision for differences $(\Delta f_{j+1})(z) = \tau(z)(\Delta f_j)(z^2)$.

M Divided differences $D_j = 2^j\Delta$. Derived scheme for divided differences:

$$\sigma(z) = \sigma^{[0]}(z) = \frac{1+z}{2}\sigma^{[1]}(z).$$

Generally, for $p > 0$ define $D_j^{[p]} = (D_j)^p$.

N If $D_j f_j$ and f_j converge uniformly to continuous functions $G(t)$ and $F(t)$ correspondingly, then $G = F'$.

O Suppose that S is affine. Let $f_{j+1} = Sf_j$.

(a) Obviously, if $\|D_j f_j\|_\infty < C\gamma^j$ then $\|\Delta f_j\|_\infty < C(\gamma/2)^j$

(b) If $\|\Delta f_j\|_\infty < C\gamma^j$ then $\|\ell_{j+1} f_{j+1} - \ell_j f_j\|_\infty < C\gamma^j$.

(c) (i) If $\|\ell_{j+1}f_{j+1} - \ell_j f_j\|_\infty < C\gamma^j$ for some fixed $0 \le \gamma < 1$ then f_j converges.
  (ii) If $\|\ell_{j+1}f_{j+1} - \ell_j f_j\|_\infty < C\gamma^j$ for some fixed $\gamma > 1$ then $\|f_j\|_\infty < C'\gamma^j$.

(d) We can combine the above three statements into a "reduction" lemma:
  (i) If $\|D_j f_j\|_\infty < C\gamma^j$ for $\gamma > 2$ then $\|f_j\|_\infty < C'(\gamma/2)^j$
  (ii) If $\|D_j f_j\|_\infty < C\gamma^j$ for $\gamma < 2$ then f_j produce a continuous function.

P Non-trivial convergent scheme needs to be affine.

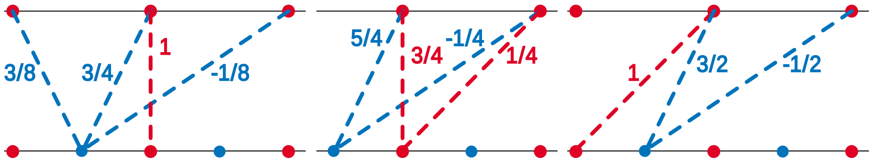

FIGURE 1. *Three point scheme and its derived schemes.*

2. EXAMPLE: "3PT SCHEME".

A Scheme interpolates three neighbors with a parabola and inserts its value at a new point which is halfway between the first and the second points. Its mask is given by

$$\sigma(z) = \frac{-1}{8}z^{-3} + \frac{3}{4}z^{-1} + 1 + \frac{3}{8}z = \frac{z^{-3}}{8}(-1 + 6z^2 + 8z^3 + 3z^4).$$

B Factoring out $(1+z)/2$ we obtain the scheme for the first derivatives:

$$\sigma^{[1]}(z) = \frac{z^{-3}}{4}(-1 + z + 5z^2 + 3z^3).$$

C Another factorization gives us the scheme for the second divided differences:

$$\sigma^{[2]}(z) = \frac{z^{-3}}{2}(-1 + 2z + 3z^2).$$

D Yet another factorization gives us the scheme for the third divided differences:

$$\sigma^{[3]}(z) = z^{-3}(-1 + 3z).$$

E Distilling the subdivision mask for the third divided differences into readable subdivision rules we get:

$$f^{[3]}_{j+1,2k} = f^{[3]}_{j,k+1};$$
$$f^{[3]}_{j+1,2k+1} = 3f^{[3]}_{j,k+1}.$$

It follows that $\|D_j^{[3]} f_j\|_\infty < C_0 3^j$ where $C_0 := \|D_0^{[3]} f_0\|_\infty$. Then $\|D_j^{[2]} f_j\|_\infty < C_1 (3/2)^j$ From reduction lemma the first derived scheme converges to a continuous function. Hence, the original 3pt scheme produces C^1 functions.

3. EVERYTHING EIGEN

A Eigenvalue λ and eigenvector e for matrix M if $Me = \lambda e$.

B Local matrix $\bar{S}$ for subdivision. There exists a finite neighborhood which determines local behavior of the scheme. Especially useful when we know behavior of the scheme everywhere but at a particular point.

C Eigenvectors and eigenvalues of $\bar{S}$ are important. Convergence at a point can be determined by them.

D Local eigenvector can be extended to be global. Given $\bar{e}$ we define e recursively extending its region via $e = S\bar{e}/\lambda$. In the end, we have $Se = \lambda e$ globally.

E If e is an eigenvector corresponding to λ for an affine subdivision scheme S then its derived scheme has an eigenvector $D_j e$ corresponding to eigenvalue 2λ if $D_j e \neq 0$. (That means that the all-ones eigenvector disappears while all others get "upgraded" to correspond to eigenvalues twice the previous one.)

Hence, the "canonic" eigenvalue structure for functional subdivision is $(1, 1/2, 1/4, \ldots, 2^{-n}, \lambda_{n+2}, \ldots)$ with $\lambda_{n+2} < 2^{-n}$ for a subdivision scheme that has n continuous derivatives. For example, for the cubic B-spline local matrix the eigenstructure is $(1, 1/2, 1/4, 1/8, 1/8)$.

F Define eigenfunction $\phi_\lambda := S^\infty[e_\lambda]$.

G Scaling relation for eigenfunctions: $\phi_\lambda(t) = \lambda\phi_\lambda(2t)$.

H Let S be a subdivision scheme producing C^n functions. Any non-zero eigenfunction ϕ_λ for $\lambda \geq 2^{-n}$ is a polynomial t^k with $\lambda = 2^{-k}$.

Indeed, consider ϕ_λ. Differentiating the refinement equation for $\phi_\lambda(t)$ we get $\phi_\lambda^{(n)}(t) = 2^n \lambda \phi_\lambda^{(n)}(2t)$. It follows that $\phi_\lambda^{(n)}(t)$ is either zero or a constant. Hence, $\phi_\lambda(t)$ is a polynomial. Looking back at the refinemen equation we have $\phi_\lambda(t) = t^k$ and $\lambda = 2^{-k}$.

I It is different for tangent continuity of curve subdivision. A useful eigenstructure can be $(1, \lambda, \ldots)$ with $\lambda < 1$.