CS 175. HOMEWORK 4: DUE 5/9/2000 BEFORE MIDNIGHT

May 3, 2000

Four-point scheme near the boundary. In this exercise we examine four-point scheme near boundary. Use a computer algebra package or MATLAB for matrix manipulations.

Consider the 4pt scheme on the positive axis $x \geq 0$. The knot sequence is given by $n_{j,k} = k2^{-j}$, and the subdivision scheme builds sequences $f_{j,k}$ where $j = 0, 1, \ldots$ and $k = 0, 1, \ldots$.

The subdivision scheme $S^{b,p}$ is given as follows:

$$\begin{aligned} f_{j+1,2k} &= f_{j,k} \quad \text{for } k \geq 0, \\ f_{j+1,2k+1} &= 1/16(-f_{j,k-1} + 9f_{j,k} + 9f_{j,k+1} - f_{j,k+2}) \quad \text{for } k \geq 1, \\ f_{j+1,1} &= P^{b,p}[f_j]; \end{aligned}$$

here $P^{b,p}$ is a special predictor used near the boundary. Thus, the scheme $S^{b,p}$ differs from the 4pt scheme only near the origin. It is known that the 4pt scheme produces C^1 functions.

A (10pts) Let $F(x)$ be the function produced by $S^{b,p}$. Show that the derivative $F'(x)$ exists and is continuous for any $x > 0$.

B (15pts) The 4pt scheme uses cubic interpolation to predict "odd" points. In order to find the expression for $P^{b,p}$ we make use of polynomial interpolation as well. Namely, let π^p be the polynomial of degree p interpolating the data on level j at the knots $n_{j,0}, \ldots, n_{j,p}$ so that

$$\begin{aligned} \pi^p(n_{j,0}) &= f_{j,0}, \\ &\ldots \\ \pi^p(n_{j,p}) &= f_{j,p}, \end{aligned}$$

and set $f_{j+1,1} = P^{b,p}[f_j] = \pi^p(n_{j+1,1})$.

In the linear case ($p = 1$), we get $f_{j+1,1} = 1/2(f_{j,0} + f_{j,1})$. Find the $P^{b,p}$ for the quadratic ($p = 2$) and cubic ($p = 3$) cases.

C (15pts) Find the 4×4 subdivision matrices $M^{b,p}$ in the neighborhood of the origin for $p = 1, 2, 3$, so that

$$\begin{bmatrix} f_{j+1,0} \\ f_{j+1,1} \\ f_{j+1,2} \\ f_{j+1,3} \end{bmatrix} = M^{b,p} \begin{bmatrix} f_{j,0} \\ f_{j,1} \\ f_{j,2} \\ f_{j,3} \end{bmatrix},$$

or $\mathbf{f}_{j+1} = M^{b,p}\mathbf{f}_j$.

Make sure that the polynomial reproduction conditions are satisfied to verify that your matrices $M^{b,p}$ are correct. In particular, that will verify that your schemes are affine, that is $\mathbf{e} = M^{b,p}\mathbf{e}$ where

$$\mathbf{e} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

D (15pts) Find eigenvalues of the matrices $M^{b,p}$. How many continuous derivatives can we expect the limit function F to have at the origin $x = 0$?

E (15pts) Define the first divided differences as follows

$$f_{j,k}^{[1]} := \frac{f_{j,k+1} - f_{j,k}}{n_{j,k+1} - n_{j,k}} = 2^j (f_{j,k+1} - f_{j,k}),$$

Define the following two matrices, the differencing matrix

$$D := \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix},$$

and the "integrating" matrix

$$Q := \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Note that $\mathbf{f}_j^{[1]} = 2^j D\mathbf{f}_j$, where $\mathbf{f}_j^{[1]} := [f_{j,0}^{[1]}, f_{j,1}^{[1]}, f_{j,2}^{[1]}]^T$; and that $\mathbf{f}_j = 2^{-j} Q\mathbf{f}_j^{[1]} + f_{j,0}\mathbf{e}$. Prove that the equation

$$\mathbf{f}_{j+1}^{[1]} = 2DM^{b,p}Q\mathbf{f}_j^{[1]},$$

governs the evolution of first divided differences.

F (10pts) Find the derived subdivision scheme 3×3 matrices $T^{b,p} := 2DM^{b,p}Q$ (as usual $p = 1, 2, 3$).

* Note that if the derived subdivision converges to a continuous function then that function is the derivative of the limit function of the original scheme (if that exists). Note that the derived schemes are no longer interpolating.

G (20pts) Find the eigenvalues of $T^{b,p}$ and find its left and right eigenvectors $\mathbf{l}^{0,p}$ and $\mathbf{r}^{0,p}$ corresponding to eigenvalue 1. Make sure that $\mathbf{r}^{0,p} = [1, 1, 1]^T$ for all $p = 1, 2, 3$, and normalize $\mathbf{l}^{0,p}$ so that ${\mathbf{l}^{0,p}}^T \mathbf{r}^{0,p} = 1$. Show that if F is the limit function of the original subdivision scheme then

$$F'(0) = {\mathbf{l}^{0,p}}^T \mathbf{f}_0^{[1]},$$

and find the corresponding linear functionals $\delta_p = {\mathbf{l}^{0,p}}^T D$, so that $F'(0) = \delta_p \mathbf{f}_0$ for $p = 1, 2, 3$ (that is, find the coefficients with which one should combine the values $f_{0,0}, f_{0,1}, f_{0,2}, f_{0,3}$ in order to compute the derivative of the limit function at the origin.)

Find left eigenvectors of the original subdivision matrix $M^{b,p}$. Which one of them is directly related to δ_p? (p=1,2,3)

H (extra 10pts) Prove that the limit function of the scheme $S^{b,3}$ is a cubic polynomial on the segment $[0, 1]$, and check that your functional δ_3 is indeed correct.