

Overcoming Limitations of Game-Theoretic Distributed Control

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Abstract—Recently, game theory has been proposed as a tool for cooperative control. Specifically, the interactions of a multi-agent distributed system are modeled as a non-cooperative game where agents are self-interested. In this work, we prove that this approach of non-cooperative control has limitations with respect to engineering multi-agent systems. In particular, we prove that it is not possible to design budget balanced agent utilities that also guarantee that the optimal control is a Nash equilibrium. However, it is important to realize that game-theoretic designs are not restricted to the framework of non-cooperative games. In particular, we demonstrate that these limitations can be overcome by conditioning each player’s utility on additional information, i.e., a state. This utility design fits into the framework of a particular form of stochastic games termed state-based games and is applicable in many application domains.

I. INTRODUCTION

Cooperative control problems entail several autonomous players seeking to collectively accomplish a global objective. Examples of cooperative control problems are numerous, e.g., the sensor coverage problem [1], [2], consensus [3], [4], power control in a wireless network [5], and network coding [6], [7]. Regardless of the specific application domain, the central goal is the same: to derive desirable collective behaviors through the design of local control algorithms.

One approach to cooperative control problems that is receiving significant attention is game-theoretic control. Specifically, the approach is to model the interactions of a multi-agent control system as a non-cooperative game where agents are “self-interested” [8], [9]. There are wide-ranging advantages to this form of a distributed architecture including robustness to failures and environmental disturbances, reducing communication requirements, improving scalability, etc. The two major challenges of modeling a multi-agent system as a non-cooperative game are (i) designing local agent objective functions, which may very well be in conflict with one another, and (ii) designing distributed learning dynamics so that the resulting global behavior is desirable with respect to the global objective.

This paper focuses on the first challenge: *utility design*. Utility design for non-cooperative control of distributed systems is a delicate task with many competing objectives. The two primary objectives are

- (i) *Existence*: A utility design should guarantee that a pure Nash equilibrium exists.

- (ii) *Efficiency*: A utility design should guarantee that all Nash equilibria are efficient with respect to the global objective.

In a non-cooperative setting where players are self-interested, a (pure) Nash equilibrium represents an individually agreeable allocation. Therefore, the existence of efficient equilibria for a utility design is of the utmost importance. There are several additional objectives for utility design that are desirable in many application domains including:

- (iii) *Locality of information*: A player’s utility should only depend on local information.
- (iv) *Tractability*: Computing the utility design should be tractable in games with a large number of players.
- (v) *Budget balance*: In many problems that involve costs, the cost must be completely absorbed by the players. For example, in a network formation problem the cost associated with building and maintaining a network must be completely distributed to the players.

To this point, utility design has primarily been approached in an application-specific manner, e.g., [5], [10]–[12]. For each application domain, the authors designed a non-cooperative game and then analyzed the desirability of the game by focusing on issues such as existence and efficiency of equilibria, budget balance, computational complexity, and locality of information. While the notion of desirability has been fairly consistent, the game-theoretic design has been strongly tied to the application domain.

Our goal in this paper is to investigate utility design in an application independent framework. To that end, we focus on the class of *distributed welfare games* (DWGs) introduced in [2]. The DWG class formalizes the notion of a resource allocation game and can model a wide variety of applications, e.g., sensor placement, wireless power management, network formation, routing, and job scheduling. In a DWG, there exists a set of resources, each with a welfare (or cost) function that depends only on the subset of players choosing that resource. A player’s utility is defined as some fraction of the welfare garnered at each resource the player selected; hence, a player’s utility is local by definition. In a DWG, the complete structure of the utilities is determined by how the global planner chooses to distribute the welfare at each resource. Based on this structure, one can explicitly study the impact of a *distribution rule* on the desirability issues mentioned above. See Section II for more background on DWGs.

Recent results have provided a few promising distribution rules that are inherited from the traditional economic cost sharing literature [13]. The designs are referred to as the *wonderful life utility* [14] and the (*weighted*) *Shapley value* [15]–[17]. These designs are promising because they both (i)

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Distribution rule	Equilibrium exists	Budget balanced	Tractable	Price of stability	Price of anarchy
Wonderful life	yes	no	yes	1	1/2
Shapley value	yes	yes	no	1/2	1/2
Priority-based	yes	yes	yes	1	1/2

TABLE I
SUMMARY OF DISTRIBUTION RULES FOR DISTRIBUTED WELFARE GAMES WITH SUBMODULAR WELFARE FUNCTIONS.

guarantee the existence of a Nash equilibrium in all DWGs and (ii) guarantee that the Price of Anarchy (PoA) is $1/2$ when the welfare functions are *submodular* [2], which is common in many resource allocation problems. In addition to guaranteeing existence and efficiency, both the Shapley value and the wonderful life utility designs result in potential games [18] which can be exploited in distributed learning algorithms, e.g., [19], [20].

However, despite the promise of the two utility designs described above, there are some fundamental limitations. The first limitation is that a budget balanced distribution rule guarantees the existence of an equilibrium for any game if and only if the distribution rule is conditioned on each player's Shapley value, which is intractable [21]. The second limitation is that many of the desirable properties described above are in conflict. The summary in Table I highlights this fact. The wonderful life utility design is tractable and guarantees a Price of Stability (PoS) of 1, however it is not budget balanced. On the other hand, the Shapley value utility design is budget balanced, but is intractable and has a PoS of $1/2$. *The first contribution of this work is to prove that there is a fundamental conflict between budget-balance and efficiency in non-cooperative utility design.* Specifically, we prove that it is impossible for a budget balanced rule to maintain a price of stability of 1 (Theorem 1). Furthermore, if a budget balanced rule guarantees the existence of an equilibrium in all games, then it must have a price of stability less than $1/2$ (Theorem 3).

The fundamental restrictions described above seem damaging to the goal of non-cooperative distributed control, however the main results of this paper illustrate that it is possible to bypass the limitations by changing the underlying structure of the game considered. Specifically, while the tools of non-cooperative game theory are valuable, there is no reason to be restricted to that setting. *The second contribution of this work is to show that by conditioning utilities on extra information (state), it is possible to design a budget balanced distribution rule that is tractable and maintains a price of stability of 1.* The key idea behind this new utility design is to change the underlying game so that it is a specific form of a stochastic game [22] termed a state-based game [23]. In this framework, we design *priority-based* rules that outperform both the wonderful life utility and the Shapley value in all attributes, as highlighted Table I.

II. BACKGROUND

A. Defining distributed welfare games

A Distributed Welfare Game (DWG), introduced in [2], is a *resource allocation game* where each player's utility is defined as some fraction of the welfare garnered. Specifically, there exists a set of players $N := \{1, \dots, n\}$ and a finite set of resources \mathcal{R} that are to be shared by the players. Each player $i \in N$ is assigned an action set $\mathcal{A}_i \subseteq 2^{\mathcal{R}}$ where $2^{\mathcal{R}}$ denotes the power sets of \mathcal{R} ; therefore, a player may have the option of selecting multiple resources. The set of joint actions is denoted by $\mathcal{A} := \mathcal{A}_1 \times \dots \times \mathcal{A}_n$. For an action profile $a = (a_1, a_2, \dots, a_n) \in \mathcal{A}$, let a_{-i} denote the profile of player actions *other than* player i , i.e., $a_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$. With this notation, we will sometimes write a profile a of actions as (a_i, a_{-i}) .

In a DWG there is a global welfare function $W : \mathcal{A} \rightarrow \mathbb{R}$ that measures the welfare associated with a particular action profile. We consider *separable* global welfare functions of the form

$$W(a) = \sum_{r \in \mathcal{R}} W^r(a^r),$$

where $W^r : 2^N \rightarrow \mathbb{R}_+$ is the welfare function for resource r and a^r denotes the subset of players that selected resource r in the joint allocation a , i.e., $a^r := \{i \in N : r \in a_i\}$. Each player is assigned a utility function $U_i : \mathcal{A} \rightarrow \mathbb{R}$ that is equal to some fraction of the welfare garnered. Specifically, a player's utility function is of the form

$$U_i(a_i, a_{-i}) = \sum_{r \in a_i} f^r(i, a^r), \quad (1)$$

where $\{f^r(1, a^r), \dots, f^r(n, a^r)\}$ defines how the welfare garnered from resource r is distributed across the players. We refer to $f := \{f^r(1, a^r), \dots, f^r(n, a^r)\}_{r \in \mathcal{R}, a^r \subseteq N}$ as the *distribution rule*. A distribution rule must satisfy the following properties: for any player $i \in N$, resource $r \in \mathcal{R}$, and player set $a^r \subseteq N$

- (i) $f^r(i, a^r) \geq 0$,
- (ii) $i \notin a^r \Rightarrow f^r(i, a^r) = 0$,
- (iii) $\sum_i f^r(i, a^r) \leq W^r(a^r)$.

We refer to distribution rules that satisfy (iii) with equality as *budget balanced distribution rules*.

The efficacy of a distribution rule is measured by whether the distribution rule guarantees both the existence and efficiency of a pure Nash equilibrium. An action profile $a^* \in \mathcal{A}$ is called a pure Nash equilibrium if for all players $i \in N$,

$$U_i(a_i^*, a_{-i}^*) = \max_{a_i \in \mathcal{A}_i} U_i(a_i, a_{-i}^*). \quad (2)$$

A pure Nash equilibrium represents a scenario for which no player has an incentive to unilaterally deviate, i.e., the resource allocation is stable. We will henceforth refer to a pure Nash equilibrium as simply an equilibrium.

We use the *price of anarchy (PoA)* and *price of stability (PoS)* to measure the efficiency of equilibria [24]. The price of anarchy gives a lower bound on the global welfare achieved by any equilibrium while the price of stability gives a lower bound on the global welfare associated with the best equilibrium. Specifically, let \mathcal{G} denote a set of DWGs. A game $G \in \mathcal{G}$ consists of the player set, N , action sets, \mathcal{A}_i , and utility functions U_i . For any particular game $G \in \mathcal{G}$ let $\mathcal{E}(G)$ denote the set of equilibria, $PoA(G)$ denote the price of anarchy, and $PoS(G)$ denote the price of stability for the game G where

$$PoA(G) := \min_{a^{ne} \in \mathcal{E}(G)} \frac{W(a^{ne})}{W(a^{opt})} \quad (3)$$

$$PoS(G) := \max_{a^{ne} \in \mathcal{E}(G)} \frac{W(a^{ne})}{W(a^{opt})}, \quad (4)$$

where $a^{opt} \in \arg \max_{a^* \in \mathcal{A}} W(a^*)$. We define the price of anarchy and price of stability for the set of DWGs \mathcal{G} as

$$PoA(\mathcal{G}) := \inf_{G \in \mathcal{G}} PoA(G), \quad (5)$$

$$PoS(\mathcal{G}) := \inf_{G \in \mathcal{G}} PoS(G). \quad (6)$$

For a more comprehensive review of the game-theoretic concepts introduced in this section, we refer the readers to [20], [24]–[26].

B. Prior work on distributed welfare games

Prior work on distributed welfare games has focused on identifying distribution rules that guarantee desirable properties. In [2], [27], the authors identify two types of distribution rules, inherited from the traditional economic cost sharing literature, that guarantee the existence of an equilibrium irrespective of the welfare functions, the number of players, or each player's respective action set. The first such distribution rule is known as the *wonderful life utility* [14]. The wonderful life utility distributes the welfare according to each player's marginal contribution, i.e.,

$$f^r(i, a^r) := W^r(a^r) - W^r(a^r \setminus i). \quad (7)$$

The wonderful life utility always guarantees that the allocation which maximizes the global welfare W is an equilibrium; hence, the price of stability is 1 when utilizing such a design. However, the wonderful life utility provides no guarantees on the amount of welfare distributed.

The second such distribution rule is known as the *Shapley value* [15]–[17]. This rule distributes the welfare according to each player's Shapley value, i.e.,

$$f^r(i, a^r) := \sum_{S \subseteq a^r: i \in S} \omega_S (W^r(S) - W^r(S \setminus i)). \quad (8)$$

where the weight of the player set S is defined as

$$\omega_S := \frac{(|a^r| - 2)! (|S| - 1)!}{|a^r|!}.$$

The Shapley value is a budget balanced distribution rule that always guarantees the existence of an equilibrium. However, the allocation that maximizes the global welfare is not guaranteed to be an equilibrium. Furthermore, computing a Shapley value is intractable for games with a large number of players.

Table I compares the properties of the wonderful life utility and the Shapley value and highlights a tension between developing distribution rules that are budget balanced, tractable and guarantee a price of stability of 1. Note that the price of stability is a particularly important measure of efficiency due to the existence of distributed learning algorithms that guarantee convergence to the best equilibrium, e.g., [28]–[30].

Note that many of the results in [2] focus on the special case where the welfare functions considered belong to an important class of welfare functions called *submodular*. Specifically, a welfare function $W^r : 2^N \rightarrow \mathbb{R}$ is submodular if

$$W^r(X) + W^r(Y) \geq W^r(X \cap Y) + W^r(X \cup Y)$$

for all $X, Y \subseteq N$. Submodularity corresponds to the notion of decreasing marginal contribution and is a very commonly observed property across resource allocation problems, e.g., [31], [32]. In the context of submodular games, it was shown in [2] that the price of anarchy of the wonderful life and Shapley value distribution rules is 1/2. In this paper, we will also often focus on submodular DWGs.

III. LIMITATIONS OF NON-COOPERATIVE DESIGNS

We are now ready to explore the feasibility of deriving desirable distribution rules for distributed welfare games. We will focus on the case of submodular DWGs and prove two theorems illustrating the impossibility of achieving all of the desirable properties mentioned thus far.

Our first result is that no budget balanced distribution rule can guarantee a price of stability of 1 in all DWGs with submodular welfare functions.

Theorem 1. *Consider the set of distributed welfare games with submodular welfare functions and a budget balanced distribution rule. The price of stability is strictly less than 1.*

Proof: Consider a DWG with players set $N = \{1, 2\}$, a budget balanced distribution rule f , and a single resource r with a welfare function of the form

$$W^r(a^r) = 1 \Leftrightarrow a^r \neq \emptyset.$$

If any player is at the resource r , then the entire welfare of 1 is garnered. Consider the allocation in which both players select r . The utility garnered to player $i \in N$ for this allocation is $f^r(i, N)$. Without loss of generalities, let $f^r(1, N) \geq f^r(2, N)$. Note that $f^r(1, N) \geq 1/2$.

Suppose player 1 has an option of selecting an alternative resource r_1 that is only available to player 1, i.e., $\mathcal{A}_1 = \{r, r_1\}$ and $\mathcal{A}_2 = \{r\}$. Resource r_1 has a welfare function W^{r_1} of the form

$$W^{r_1}(a^{r_1}) = f^r(1, \{1, 2\}) - \epsilon \Leftrightarrow a^{r_1} \neq \emptyset,$$

for some $\epsilon > 0$. Since f is a budget balanced distribution rule, it is easy to show that the profile $a^{\text{ne}} = (r, r)$ is the unique equilibrium for any $\epsilon > 0$. The optimal allocation is the profile $a^{\text{opt}} = (r_1, r)$ which garners a total welfare of

$$\begin{aligned} W(a^{\text{opt}}) &= 1 + f^r(1, N) - \epsilon, \\ &\geq \frac{3}{2} - \epsilon. \end{aligned}$$

Therefore, since n and ϵ are arbitrary, this gives us a price of stability of $2/3 < 1$. ■

Notice that the example in the proof of Theorem 1 proves that one cannot guarantee a price of stability greater than $2/3$ using a budget balance distribution rule.

Moving to our second impossibility result, we will now show that if we would like a budget balanced distribution rule that also guarantees the existence of an equilibrium in all submodular DWGs, the price of stability is at most $1/2$.

To prove this result we will first restrict our attention to *valid* distribution rules. Roughly speaking, a valid distribution ensures that the fraction of the welfare garnered by a particular player diminishes as the player set grows.

Definition 1 (Valid Distribution Rules). *We call a distribution rule f valid if for any player sets $X \subseteq Y \subseteq N$ and resource $r \in \mathcal{R}$ such that $W^r(X) = W^r(Y)$, then the fraction of welfare distributed to any player $i \in X$ satisfies*

$$f^r(i, X) \geq f^r(i, Y).$$

Lemma 2. *Consider the set of distributed welfare games with submodular welfare functions and a valid budget balanced distribution rule. The price of stability is $\leq 1/2$.*

Proof: The proof uses an extension of the proof of Theorem 1. Consider a DWG with player set $N = \{1, \dots, n\}$, a valid distribution rule f , and a single resource r with a welfare function of the form

$$W^r(a^r) = 1 \Leftrightarrow a^r \neq \emptyset.$$

If any player is at the resource r , then the entire welfare of 1 is garnered. Consider the allocation in which all players select r . The utility garnered to player i for this allocation is $f^r(i, N)$. Without loss of generalities, let $f^r(1, N) \geq f^r(2, N) \geq \dots \geq f^r(n, N)$. Note that $f^r(n, N) \leq 1/n$.

Suppose each player $i \in \{1, \dots, n-1\}$ has an option of selecting an alternative resource r_i that is only available to that particular player, i.e., $\mathcal{A}_i = \{r, r_i\}$ for all players $i \in \{1, \dots, n-1\}$ and $\mathcal{A}_n = \{r\}$. Each resource r_i has a welfare function W^{r_i} of the form

$$W^{r_i}(a^{r_i}) = f^r(i, N) - \epsilon \Leftrightarrow a^{r_i} \neq \emptyset,$$

for some $\epsilon > 0$. Since f is a valid budget balanced distribution rule, it is easy to show that the profile $a^{\text{ne}} = (r, \dots, r)$ is the unique equilibrium for any $\epsilon > 0$. The optimal allocation is the profile $a^{\text{opt}} = (r_1, r_2, \dots, r_{n-1}, r)$ which garners a total

welfare of

$$\begin{aligned} W(a^{\text{opt}}) &= 1 + \sum_{i=1}^{n-1} (f^r(i, N) - \epsilon), \\ &\geq \frac{2n-1}{n} - (n-1)\epsilon. \end{aligned}$$

Therefore

$$\lim_{\epsilon \rightarrow 0} \frac{W(a^{\text{ne}})}{W(a^{\text{opt}})} \leq \frac{n}{2n-1}.$$

Since n is arbitrary, this gives us a price of stability of $1/2$. ■

Though the above lemma only applies in the context of valid distribution rules, all natural distribution rules are valid. For example, the (weighted) Shapley distribution rule is valid. Further, recent results in [21] have shown that any budget balanced distribution rule that guarantees the existence of an equilibrium must be a (weighted) Shapley value, which is intractable to compute. This result was proven in the context of network formation games, but can be shown to hold for DWGs using a parallel proof that is omitted due to lack of space. Combining this result with the above lemma yields the following theorem.

Theorem 3. *Consider the set of distributed welfare games with submodular welfare functions and a budget balanced distribution rule that guarantees the existence of an equilibrium in all games. The price of stability is $\leq 1/2$.*

IV. STATE-BASED NON-COOPERATIVE DESIGNS

The framework of non-cooperative distributed control provides a promising paradigm for resource allocation; however, the preceding section demonstrates two fundamental limitations. In general, designing local utility functions that are budget balanced and guarantee the existence of an equilibrium requires computing a Shapley value for each player, which is often computationally intractable. Further, it is impossible for a budget balanced distribution rule to guarantee a price of stability greater than $1/2$. In this section we seek to overcome these limitations by conditioning a player's utility on additional information.

In many settings, players' utility functions are directly influenced by an exogenous state variable. In this section, we consider the framework of *state-based games* introduced in [23] which generalizes the non-cooperative game setting to such an environment. state-based games are a simplification of the class of stochastic games [22]. In a state-based game, there exists a finite state space X . Each player $i \in N$ has an action set \mathcal{A}_i and a state dependent utility function $U_i : \mathcal{A} \times X \rightarrow \mathbb{R}$. We assume that the state evolves according to a state-transition function $P : \mathcal{A} \times X \rightarrow \Delta(X)$ where $\Delta(X)$ denotes the set of probability distributions over the finite state space X .

A state-based game proceeds as follows. Let the state at time $t \in \{0, 1, \dots\}$ be denoted by $x(t) \in X$. At any time t , each player i selects an action $a_i(t) \in \mathcal{A}_i$ randomly based on available information. The state $x(t)$ and the action profile $a(t) := (a_1(t), \dots, a_n(t))$ together determine each player's cost

$U_i(a(t), x(t))$ at time t . Each player selects an action $a_i(t)$ simultaneously seeking to maximize his one-stage expected utility $\mathbf{E}[U_i(a(t), x(t))]$, where the expectation is over player i 's belief regarding the action choice of the other players, i.e., $a_{-i}(t)$. In this case, a player's strategy is unaffected by how his current action impacts the state dynamics and potential future rewards. After each player selects his respective action, the ensuing state $x(t+1)$ is chosen randomly according to the probability distribution $P(a(t), x(t)) \in \Delta(X)$. In this paper, we restrict our attention to state dynamics that satisfy

$$a(t) = a(t-1) \Rightarrow x(t+1) = x(t). \quad (9)$$

This paper focuses on analyzing equilibrium behavior in such games. We consider state-based Nash equilibria, which generalize pure Nash equilibria to this state-based setting [23].

Definition 2 (state-based Nash Equilibrium). *The action state pair $[a^*, x^*]$ is a state-based Nash equilibrium if for every player $i \in N$ and every state x' in the support of $P(a^*, x^*)$*

$$U_i(a_i^*, a_{-i}^*, x') = \max_{a_i \in \mathcal{A}_i} U_i(a_i, a_{-i}^*, x').$$

If $[a^*, x^*]$ is a state-based Nash equilibrium, then no player $i \in N$ will have a unilateral incentive to deviate from a_i^* provided that all other players play a_{-i}^* regardless of the state that emerges according to the transition function $P(a^*, x^*)$. We use the term equilibrium to mean state-based Nash equilibrium in the discussion that follows.

Given a state-based game, an equilibrium may or may not exist. We consider a simplified framework of *state-based potential games*, introduced in [23], for which an equilibrium is guaranteed to exist. state-based potential games generalize potential games [18] to the state-based setting.

Definition 3 (state-based Potential Games). *A state-based game with state transition function P is a state-based potential game if there exists a potential function $\phi: \mathcal{A} \rightarrow \mathbb{R}$ such that for any action state pair $[a, x] \in \mathcal{A} \times X$, player $i \in N$, and action $a'_i \in \mathcal{A}_i$*

$$U_i(a'_i, a_{-i}, x) - U_i(a, x) > 0 \Rightarrow \phi(a'_i, a_{-i}) - \phi(a) > 0.$$

This condition states that players' cost functions are aligned with the potential function. To see that an equilibrium exists in any state-based potential game, let $[a^*, x^*]$ be any action state pair such that $a^* \in \arg \min_{a \in \mathcal{A}} \phi(a)$. The action state pair $[a^*, x^*]$ is an equilibrium.

We measure the efficiency of an equilibrium by extending the measures of the price of anarchy and price of stability to the state-based setting. Specifically, let \mathcal{G} denote the set of state-based games and let $W: \mathcal{A} \times X \rightarrow \mathbb{R}$ be a state-based welfare function. For any particular game $G \in \mathcal{G}$ let $\mathcal{E}(G)$ denote the set of equilibria, i.e., $\mathcal{E}(G) := \{[a, x] \in \mathcal{A} \times X : [a, x] \text{ is an equilibrium of the game } G\}$. The price of anarchy and price of stability of the state-based game G now take on

the form

$$PoA(G) := \min_{[a^{ne}, x^{ne}] \in \mathcal{E}(G)} \frac{W(a^{ne}, x^{ne})}{W(a^{opt}, x^{opt})} \quad (10)$$

$$PoS(G) := \max_{[a^{ne}, x^{ne}] \in \mathcal{E}(G)} \frac{W(a^{ne}, x^{ne})}{W(a^{opt}, x^{opt})} \quad (11)$$

where $[a^{opt}, x^{opt}] \in \arg \max_{[a^*, x^*] \in \mathcal{A} \times X} W(a^*, x^*)$. The price of anarchy and price of stability for the set of DWGs \mathcal{G} is then defined as in (5) and (6).

V. A PRIORITY-BASED DISTRIBUTION RULE

Moving from traditional non-cooperative designs to state-based designs gives an additional degree of freedom when designing distribution rules. The extra degree of freedom is enough to address the limitations we identified in Section III. In particular, in this section we provide the design of a state-based distribution rule that guarantees the existence of an equilibrium, maintains a price of stability of 1, is tractable, and is budget balanced.

Before discussing the details of the distribution rule, we provide a brief sketch of the main idea. Suppose at each resource there is an ordering, or priority, for the players utilizing that resource. We condition our distribution rule on this priority in the following way: players are placed one by one at the resource in order of their priority and the welfare distributed to a particular player is set as the player's marginal contribution when the player joined the resource. Therefore, players with lower priority have no impact on the player's received welfare. Utilizing the framework of state-based game to facilitate this distribution rule requires defining a state space that reflects this notion of priority and defining a state transition function that specifies how the priorities are affected by changes in strategies.

Now, we can more formally introduce the state-based distribution rule. Let X be defined as a set of states that identifies priorities at all resources. For a given allocation $a \in \mathcal{A}$, define the set of admissible states as $X(a) \subset X$ where $X(a)$ is nonempty and a state $x \in X(a)$ defines for each resource $r \in \mathcal{R}$ an order of priority for the players that selected that resource in the allocation a . The order of priority for each resource $r \in \mathcal{R}$ is described by a queue denoted by x^r where x_i^r designates the priority of player i at resource r . Any state $x \in X(a)$ satisfies the following properties for all players $i \in N$ and resources $r \in \mathcal{R}$: (i) if $r \notin a_i$, then $x_i^r = \emptyset$, (ii) if $r \in a_i$, then $x_i^r \in \{1, \dots, |a^r|\}$ where $|a^r|$ is the number of players using resource r , and (iii) $x_i^r \neq x_j^r$ for any players $i, j \in a^r$. We adopt the convention that $x_i^r = 1$ indicates that player i has the top priority at resource r . If $x_i^r < x_j^r$, we say that i has higher priority than j at resource r .

We now define the state transition function. Let $a(t-1)$ and $x(t)$ be the action profile and state at time $t-1$ and t . If one player changes his action, i.e., $a(t) = (a'_i, a_{-i}(t-1))$ for some player i , the state evolves deterministically according to the following rules:

- (i) If player i leaves resource r , i.e., $r \in a_i(t-1)$ but $r \notin a_i(t)$, then each player in the queue behind him moves

forward one spot in the queue, i.e., $x_j^r(t) < x_i^r(t) \Rightarrow x_j^r(t+1) = x_j^r(t)$ and $x_j^r(t) > x_i^r(t) \Rightarrow x_j^r(t+1) = x_j^r(t) - 1$.

- (ii) If player i joins resource r , i.e., $r \notin a_i$ but $r \in a'_i$, then player i has the lowest priority at resource r , i.e., $x_j^r(t+1) = x_j^r(t)$ for all $j \neq i$ and $x_i^r(t+1) = |a^r| + 1$.
- (iii) Otherwise the priority of players at resource r is unchanged.

If multiple players seek to join a resource simultaneously, the order of the entering players is randomly chosen. Note that $x(t+1) \in X(a(t))$. The state dynamics satisfy (9). We refer to these state dynamics as *first in first out (FIFO)*.

Before explicitly defining each player's state dependent utility function we introduce some notation. Let

$$\bar{x}_i^r := \{j \in N : x_j^r \leq x_i^r\}$$

represent the set of players at resource r that have a higher priority than player i given the state x . For any admissible action state pair $[a, x] \in \mathcal{A} \times X(a)$, the welfare distributed to player i , defined as $V_i : \mathcal{A} \times X(a) \rightarrow \mathbb{R}$, is precisely

$$V_i(a, x) = \sum_{r \in a_i} (W^r(\bar{x}_i^r) - W^r(\bar{x}_i^r \setminus i)) \quad (12)$$

For any admissible action state pair $[a, x] \in \mathcal{A} \times X(a)$, the utility of player i for any action $a' \in \mathcal{A}$ is defined as

$$U_i(a', x) = \mathbf{E}_{P(a', x)} V_i(a', x'), \quad (13)$$

where the expectation is with regard to the ensuing state x' which is chosen randomly according to the measure $P(a', x)$. Note that if $a' = (a'_i, a_{-i})$, the state transition is deterministic and the expectation can be dropped. It is important to highlight two important features of this utility design. First, this design satisfies properties (i)-(iii) of distribution for DWGs and is also budget balanced. Secondly, this design is tractable. Each player only needs to calculate his marginal contribution to a particular player set. We call this form of a distribution rule *priority-based*.

Theorem 4. *Consider any distributed welfare game with submodular welfare functions, priority-based utility functions as in (13), and FIFO state dynamics. The resulting game is a state-based potential game with potential function W and a price of stability of 1.*

Proof: Let $[a, x] \in \mathcal{A} \times X(a)$ be any admissible action state pair. Since our welfare function is submodular, we have that for any player i and resource r

$$W^r(\bar{x}_i^r) - W^r(\bar{x}_i^r \setminus i) \geq W^r(a^r) - W^r(a^r \setminus i).$$

Therefore, a player's utility is greater than or equal to his marginal contribution to the global welfare, i.e.,

$$\begin{aligned} U_i(a, x) &= V_i(a, x), \\ &= \sum_{r \in a_i} (W^r(\bar{x}_i^r) - W^r(\bar{x}_i^r \setminus i)), \\ &\geq \sum_{r \in a_i} W^r(a^r) - W^r(a^r \setminus i) \\ &= W(a) - W(\emptyset, a_{-i}). \end{aligned}$$

Suppose $U_i(a'_i, a_{-i}, x) > U_i(a, x)$ for some $a'_i \in \mathcal{A}_i$. Let $a' := (a'_i, a_{-i})$. First note that $U_i(a', x) = V_i(a', x')$ where x' is chosen according to $P(a', x)$. We seek to bound the difference in utility using a two step transition $a \rightarrow a^0 := (a_i^0, a_{-i}) \rightarrow a'$ where $a_i^0 := a'_i \cap a_i$. We first focus on bounding the term $U_i(a, x) - U_i(a^0, x)$. Note that $U_i(a^0, x) = V_i(a^0, x^0)$ where x^0 is chosen according to $P(a^0, x)$. We can bound the utility difference as

$$\begin{aligned} U_i(a, x) - U_i(a^0, x) &= \sum_{r \in a_i \setminus a_i^0} (W^r(\bar{x}_i^r) - W^r(\bar{x}_i^r \setminus i)), \\ &\geq \sum_{r \in a_i \setminus a_i^0} (W^r(a^r) - W^r(a^r \setminus i)), \\ &= W(a) - W(a^0). \end{aligned}$$

Focusing on the second term, we have

$$\begin{aligned} U_i(a', x) - U_i(a^0, x) &= U_i(a', x^0) - U_i(a^0, x^0), \\ &= \sum_{r \in a'_i \setminus a_i^0} (W^r(a^r \cup i) - W^r(a^r)), \\ &= W(a') - W(a^0). \end{aligned}$$

Combining the above two bounds, we obtain

$$U_i(a', x) - U_i(a, x) \leq W(a') - W(a).$$

This implies that

$$U_i(a', x) - U_i(a, x) > 0 \Rightarrow W(a') - W(a) > 0.$$

Finally, since the potential function of the game is W , it is clear that any allocation that maximizes W is an equilibrium. Thus, the price of stability is 1. ■

In addition to the above theorem, it is straightforward to verify that the priority-based distribution rule satisfies the conditions for a utility game set forth in [31]; therefore, for submodular welfare functions the priority-based design results in price of anarchy greater than or equal to 1/2. A comparison between the wonderful life utility, the Shapley value, and the priority-based design is given in Table I. The priority-based distribution rule achieves the desirable properties of both the wonderful life utility and the Shapley value in a computationally tractable fashion.

It is worth noting that the priority-based distribution rules can be utilized even in situations where the welfare functions are not submodular; however, the state dynamics (FIFO) will need potentially have to change to provide similar guarantees.

VI. CONCLUDING REMARKS

This paper focuses on how to design utility functions for multi-agent systems when the interactions are modeled as a non-cooperative game. The results in the paper highlight that there are fundamental limitations of utility design in the non-cooperative framework. In particular, it is impossible for a budget balanced utility design to guarantee the existence of an equilibrium in all games and to have a price of stability larger than 1/2.

However, there is no fundamental reason to limit game-theoretic designs to the non-cooperative setting. In particular, we show that by conditioning utilities on extra state it is possible to design a budget balanced distribution rule that is tractable and maintains a price of stability of 1. The key idea behind this new utility design is to change the underlying game so that it is a specific form of a stochastic game termed a state-based game.

The results in this paper present a promising new direction for utility design in non-cooperative control. In particular, the state-based utility design presented here is only one possible alternative, and a deeper study of the space of state-based utilities is clearly warranted. Further, this paper focuses entirely on the question of utility design. Another important question is how to design distributed learning algorithms that will converge to an equilibrium. This question has only begun to be addressed in the context of state-based games [23].

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