An Improved Upper Bound for the Pebbling Threshold of the n-Path

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Abstract

Given a configuration of t indistinguishable pebbles on the n vertices of a graph G, we say that a vertex v can be reached if a pebble can be placed on it in a finite number of "moves". G is said to be pebbleable if all its vertices can be thus reached. Now given the n-path P_n how large (resp. small) must t be so as to be able to pebble the path almost surely (resp. almost never)? It was known that the threshold $th(P_n)$ for pebbling the path satisfies $n2^{c\sqrt{\lg n}} \leq th(P_n) \leq n2^{2\sqrt{\lg n}}$, where $\lg = \log_2$ and $c < 1/\sqrt{2}$ is arbitrary. We improve the upper bound for the threshold function to $th(P_n) \leq n2^{d\sqrt{\lg n}}$, where $d > 1$ is arbitrary.

1 Introduction

Given a configuration of t indistinguishable markers, or *pebbles*, on the n vertices of a graph G, we define a *move* to be an operation by which two pebbles are removed from a vertex – with one of these pebbles being removed from the graph configuration, and with the other being placed on an adjacent vertex. We say that a vertex v can be *reached* if a pebble can be placed on it in a finite number of moves. Finally, we say that G is *pebbleable* if all its vertices can be thus reached. Note that we restore the graph configuration to its initial state after successive vertices are reached (or determined to be unreachable).

Graph pebbling was used by Lagarias and Saks to provide a solution to a number-theoretic question posed by Erdős and Lemke, for which the original solution was provided by Kleitman and Lemke [6]. The *pebbling number* $\pi(G)$ of a graph G is the smallest number of pebbles that need to be placed on its vertices so that G is pebbleable no matter what configuration of pebbles is used. If $G = Q_d$, the d-cube, then it is evident that placing $2^d - 1$ pebbles on the vertex antipodal to the origin leads to an unpebbleable graph, so that $\pi(Q_d) \geq 2^d$. In a landmark paper, Chung [3] proved that the pebbling number of Q_d was indeed equal to 2^d. The survey paper of Hurlbert [5] contains a wealth of results – as well as history and background – on the problem of graph pebbling.

In 1999, in the first version of [4], Czygrinow et al. introduced the notion of *pebbling thresholds*. In other words, they asked the following question: Given a graph G how large (resp. small) must the number of pebbles t be so as to be able to pebble G almost surely (resp. almost never)? We assume here (although other models can certainly be considered) that the configuration of the t pebbles on the n vertices follows the so-called Bose-Einstein distribution from statistical physics, i.e. that the $\binom{n+t-1}{t}$ configurations are all equally likely to be realized. Also, we remind the reader that a graph property is said to hold *almost surely* (resp. *almost never*) if it is satisfied with probability tending to unity (resp. zero) as the size of the problem grows to infinity. Define the *pebbling threshold function* of a graph $G = G_n$ (assuming it exists, which is not guaranteed) to be a level $th(G_n)$ such that

$$t \ll th(G_n) \Rightarrow \mathbb{P}(G \text{ is pebbleable}) \to 0 \ \ (n \to \infty),$$

and

$$t \gg th(G_n) \Rightarrow \mathbb{P}(G \text{ is pebbleable}) \to 1 \ \ (n \to \infty),$$

where, given two non-negative sequences $f = f_n$ and $g = g_n$, we write $f \ll g$ (or $g \gg f$) if $f/g \to 0$ as $n \to \infty$. Note that the threshold function is not unique if it exists; if f_n is a threshold function, then so is $K_n f_n$ for any bounded sequence K_n. *We will explicitly assume, throughout this paper, that* $K_n = 1\ \forall n$. Czygrinow et al. [4] studied the threshold function $th(G)$ for several families of graphs, including complete graphs, paths, cycles, stars, wheels, and cubes. Definitive pebbling thresholds were obtained in several cases – with the notable exceptions being the n-path P_n and the d-cube Q_d. It was exhibited, for example, that *assuming that* $th(P_n)$ *exists,* it must satisfy, in the standard computer science notation,

$$th(P_n) = \Omega(n)\ \ \&\ th(P_n) = o(n^{1+\varepsilon}), \tag{1}$$

i.e.,

$$th(P_n) \geq n$$

and for each $\varepsilon > 0$,

$$th(P_n) \ll n^{1+\varepsilon}.$$

Bekmetjev et al. [2] improved (1), when they proved that the pebbling threshold for P_n existed and satisfied

$$th(P_n) = O(n2^{2\sqrt{\lg n}})\ \&\ th(P_n) = \Omega(n2^{c\sqrt{\lg n}}),$$

where $\lg = \log_2$, and $c < 1/\sqrt{2}$ is arbitrary. The methods used in [2] were deep, and the authors recovered the threshold existence result for paths (and *all* graph sequences for that matter) from a multiset analog of Lovász's version of the Kruskal Katona theorem (see [2] for details and references). During the Summer of 2000, REU students Salzman and Wierman were able to use elementary techniques to improve the upper bound on the threshold to

$$th(P_n) = O(n2^{c\sqrt{\lg n}}),$$

where $c > \sqrt{2}$ is arbitrary, while Jablonski, a member of the ETSU REU team from the Summer of 2001, further refined the methods of Salzman and Wierman to prove that for any $d > 1$

$$th(P_n) = O(n2^{d\sqrt{\lg n}}).$$

It is this result that we present in this paper. Denote by X_j the number of pebbles on vertex j of the n-path. The summability condition

$$\sum_{j=1}^{n} \frac{X_j}{2^{j-1}} \geq 1 \tag{2}$$

for the reachability of vertex 1, noted in [4], is critical to our derivation. It arises in other contexts: In [1], pp. 236–237, Alon and Spencer analyze the so-called *tenure game* between a benevolent Department Chair, Paul, and the meanspirited Carole, the University Provost. In this game, Paul wins iff a condition similar to (2) is satisfied. Also in [1], p. 11, the Kraft inequality for prefix-free codes, and the Kraft-McMillan inequality for uniquely decipherable codes, are both seen to be versions of the inequality (2). In fact, we feel that further refinements of our result might result only if one is able to successfully exploit connections of this kind, perhaps even using results and inequalities for weighted sums of exchangeable random variables.

2 Improved Upper Bounds

Let C_n be the n-cycle. We first show

Theorem 1 *For any $d > 1$,*

$$th(C_n) = O(n2^{d\sqrt{\lg n}}).$$

Proof It is evident that for each even k,

$$\mathbb{P}(C_n \text{ is not pebbleable}) \leq n\mathbb{P}(\text{vertex 1 is unreachable}) \leq n\mathbb{P}(A_k),$$

where

$$A_k = \left\{X_1 = 0; X_2, X_n \in \{0,1\}; \ldots; X_{k/2}, X_{n+2-(k/2)} \in \{0,1,\ldots,2^{(k/2)-1}-1\}\right\}.$$

It follows that

$$\mathbb{P}(C_n \text{ is not pebbleable}) \leq n2^{2(1+2+\ldots+((k/2)-1))}\pi_k,$$

where π_k is the largest probability of a configuration in A_k. We first prove the following

Lemma 2 $\pi_k = \frac{\binom{n-k+t}{t}}{\binom{n+t-1}{t}}$.

Proof Let $J = \{1, 2, \ldots, k/2, n, n-1, \ldots, n+2-(k/2)\}$. For fixed n, t, k, we have from the fact that pebbles are distributed in the Bose-Einstein fashion,

$$\mathbb{P}(X_j = x_j; j \in J) = \frac{\binom{n+t-k-\sum x_j}{t-\sum x_j}}{\binom{n+t-1}{t}}.$$

But $\binom{a-x}{b-x}$ is a decreasing function of $x \geq 0$, so that

$$\mathbb{P}(X_j = x_j; j \in J) \leq \frac{\binom{n+t-k}{t}}{\binom{n+t-1}{t}} = \pi_k.$$

This proves the lemma.

Continuing with the proof of the theorem, we see that

$$\begin{aligned}
\mathbb{P}(C_n \text{ is not pebbleable}) &\leq n2^{((k/2)-1)(k/2)} \frac{\binom{n-k+t}{t}}{\binom{n+t-1}{t}} \\
&\leq n2^{k^2/4} \frac{\binom{n+t-k}{t}}{\binom{n+t-1}{t}} \\
&= n2^{k^2/4} \frac{n-k+1}{n+t-k+1} \cdots \frac{n-1}{n+t-1} \\
&\leq n2^{k^2/4} \left(\frac{n}{n+t}\right)^{k-1} \\
&\leq n2^{k^2/4} \left(\frac{n}{t}\right)^{k-1}. \qquad (3)
\end{aligned}$$

Setting $t = n\lceil 2^{A\sqrt{\lg n}}\rceil$ and $k = 2\lfloor B\sqrt{\lg n}/2\rfloor$ (for A and B to be determined), we see that (3) yields

$$\begin{aligned}
\mathbb{P}(C_n \text{ is not pebbleable}) &\leq n2^{\frac{B^2 \lg n}{4}} \left(\frac{1}{2^{A\sqrt{\lg n}}}\right)^{B\sqrt{\lg n}-3} \\
&= 2^{\lg n(1+(B^2/4)-AB+o(1))}. \qquad (4)
\end{aligned}$$

We wish to choose A, B such that $1 + (B^2/4) - AB < 0$ so that we will, by (4), have $\mathbb{P}(C_n$ is not pebbleable$) \to 0$. We thus need $AB - (B^2/4) > 1$, or $A > (1/B) + (B/4)$. Since we need to choose the smallest A possible, we consider the function $f(B) = (1/B) + (B/4)$, which is minimized when $B = 2$. Accordingly, we set $B = 2$ so that any $A > 1$ will yield $\mathbb{P}(C_n$ is not pebbleable$) \to 0$. This proves the theorem.

We next consider the n-path P_n, for which the key concern is the elimination of "edge effects" – whose absence makes the proof of Theorem 1 relatively straightforward.

Theorem 3 *For any* $d > 1$,

$$th(P_n) = O(n2^{d\sqrt{\lg n}}).$$

Proof Let $M = 2\lfloor\sqrt{\lg n}\rfloor$. We then note that

$$\begin{aligned}
\mathbb{P}(P_n \text{ is not pebbleable}) &= \mathbb{P}\left(\cup_{j=1}^{n} j \text{ is not reachable}\right) \\
&= \mathbb{P}(A \cup B \cup C) \\
&\leq \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) \\
&= 2\mathbb{P}(A) + \mathbb{P}(B), \qquad (5)
\end{aligned}$$

where

$$A = \left(\cup_{j=1}^{M} j \text{ is not reachable}\right),$$

$$B = \left(\cup_{j=M+1}^{n-M} j \text{ is not reachable}\right),$$

and

$$C = \left(\cup_{j=n-M+1}^{n} j \text{ is not reachable}\right).$$

Next observe that by Theorem 1

$$\begin{aligned}
\mathbb{P}(B) &\leq \mathbb{P}\left(\cup_{j=M+1}^{n-M} j \text{ is not reachable using pebbles at distance } \leq M\right) \\
&\leq \sum_{j=M+1}^{n-M} \mathbb{P}\left(j \text{ is not reachable using pebbles at distance } \leq M\right) \\
&\leq n\mathbb{P}\left(M+1 \text{ is not reachable using pebbles at distance } \leq M\right) \\
&\to 0 \quad \left(t \gg n2^{d\sqrt{\lg n}}; d > 1\right) \qquad (6)
\end{aligned}$$

We now need to consider the behaviour of $\mathbb{P}(A)$. First observe that

$$\begin{aligned}
\mathbb{P}(A) &\leq \mathbb{P}\left(\cup_{j=1}^{M} j \text{ is not reachable using pebbles at distance } \leq M\right) \\
&\leq \sum_{j=1}^{M} \mathbb{P}\left(j \text{ is not reachable using pebbles at distance } \leq M\right) \\
&\leq \sum_{j=1}^{M} \mathbb{P}\left(j \text{ is not right} - \text{reachable using pebbles at distance } \leq M\right) \\
&= M\mathbb{P}\left(1 \text{ is not right} - \text{reachable using pebbles at distance } \leq M\right) \\
&= M\mathbb{P}(D), \text{ say.} \qquad (7)
\end{aligned}$$

Now as in the proof of Theorem 1, for $k \leq M$

$$\mathbb{P}(D) \leq \mathbb{P}(X_1 = 0; X_2 \in \{0, 1\}; \ldots; X_k \in \{0, 1, \ldots, 2^{k-1} - 1\})$$

$$
\begin{aligned}
&\leq 2^{k^2/2} \max_{x_j} \mathbb{P}(X_j = x_j; 1 \leq j \leq k) \\
&\leq 2^{k^2/2} \frac{\binom{n+t-k-1}{t}}{\binom{n+t-1}{t}} \\
&\leq 2^{k^2/2} \left(\frac{n}{t}\right)^k,
\end{aligned}
$$

so that by (7),

$$\mathbb{P}(A) \leq 2\sqrt{\lg n} 2^{k^2/2} \left(\frac{n}{t}\right)^k.$$

With $t = n\lceil 2^{d\sqrt{\lg n}} \rceil; d > 1$ we thus get with, e.g., $k = \lfloor \sqrt{\lg n} \rfloor$,

$$\mathbb{P}(A) \leq 2\sqrt{\lg n} 2^{k^2/2} \left(\frac{1}{2^{d\sqrt{\lg n}}}\right)^k \to 0. \tag{8}$$

The theorem follows on combining (8) with (5) and (6).

Open Problem The overriding open problem that emerges from this paper is that of determining the exact constant in the threshold for the pebbling of the n-path. We continue to investigate this and related problems that might extend the validity and applicability of our technique.

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