On the Efficiency of Networked Stackelberg Competition

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Abstract—We study the impact of strategic anticipative behavior in networked markets. We focus on the case of electricity markets and model the market as a game between a system operator (market maker) and generators at different nodes of the network. Generators submit quantity bids and the system operator balances demand and supply over the network subject to transmission constraints. We compare the efficiency of a networked Stackelberg equilibrium, where generators anticipate the market clearing actions of the system operator, with a networked Cournot equilibrium, where generators are not anticipative. In addition to general existence results, for a 2-node network with no transmission constraints, we show that the efficiency loss is unbounded in the worst case when generators are not anticipative but no more than 1/4 when they are anticipative.

I. INTRODUCTION

Classical oligopoly models of competition have focused on a single marketplace, where the identity of participants has no impact on prices, outcomes, etc. However, marketplaces are typically much more complex and, often, the options available to individual participants are varied and highly constrained. Thus, the marketplace is not really a single market, but is instead more naturally characterized as a network of interconnected markets, with constraints on the graph of feasible exchanges.

Recently, the study of such “networked marketplaces” has garnered significant attention across economics, electrical engineering, and computer science, as a result of the importance of such models for the study of applications like electricity markets, financial markets, intermediaries, exchanges, etc.

This wide range of applications has yielded a variety of models capturing different forms of competition in networked marketplaces. The two most prominent such models are networked Bertrand competition, e.g., [1]–[3], and networked Cournot competition, e.g., [4]–[6].

For our purposes, the key motivating application is electricity markets. While the operation of electricity markets is highly complex, one simple model that has become popular for understanding market power issues is networked Cournot competition, e.g., [6]–[13]. The popularity of the networked Cournot framework stems from the fact that it allows the incorporation of Kirchoff’s laws, which constrain the allocation of generation to demand, while still yielding to analytic study.

A. Contributions of this paper

Our focus in this paper is not on Cournot competition. Rather, the goal of this paper is to initiate the study of a different form of competition in a networked marketplace – networked Stackelberg competition.

Our motivation for studying networked Stackelberg competition comes from the observation that it is common for networked marketplaces to have a market maker who is in charge of facilitating the balance of supply and demand. The need for a market maker stems from the inefficiency inherent in networked marketplaces. For example, the Independent System Operator (ISO) plays the role of the market maker for electricity markets, e.g., see [14], [15]. Similarly, exchanges typically are run by a market maker, e.g., NYSE, Yahoo Ad Exchange, NGX, CME, etc.

In the presence of a market maker, it is natural for players to anticipate the outcome of the market clearing rule implemented. This anticipation leads one away from Bertrand or Cournot competition toward Stackelberg competition.

Our main results seek to understand the impact of anticipatory behavior in networked marketplaces. To that end, we focus on a contrast between networked Cournot competition (non-anticipatory behavior) and networked Stackelberg competition (anticipatory behavior) and derive comparisons with respect to both existence and efficiency of equilibria.

To keep the study grounded, we focus on a particular networked marketplace – electricity markets – where there is a clear market maker – the ISO. In this context, we show that networked Cournot competition has an unbounded price of anarchy even in the case of two-node unconstrained networks (Proposition 1), while networked Stackelberg has a price of anarchy that is bounded by a small constant in many cases (see Theorems 1 and 2). However, an equilibrium always exists in the network Cournot model, even in general, constrained networks; whereas an equilibrium need not exist in the networked Stackelberg model, even if the network is unconstrained. Thus, anticipatory behavior leads to more efficient market outcomes, but can also lead to market failure.

† The price of anarchy is defined as the worst-case ratio of an equilibrium welfare to the social optimal welfare.
B. Related Literature

Our focus in this paper is on the impact of anticipatory behavior in networked marketplaces with a market maker, and we focus specifically on the case of electricity markets. While the networked Stackelberg model we consider is novel, the networked Cournot model we use for comparison has a long history, both in the context of electricity markets and beyond.

Classical cournot competition (without a network) dates back to the nineteenth century to the work of Cournot himself [16]. This competition model and its analysis can be found in standard texts in microeconomics, e.g., [17]. Of recent interest has been the study of efficiency loss due to strategic behavior in this framework [18]–[20]. In particular, these papers derive bounds on the efficiency loss under suitable assumptions on the inverse demand functions and the cost functions of various participants which we apply in our analysis of networked Stackelberg competition.

Cournot competition over a networked market has been studied both in the context of electricity markets [8], [10] and more generally, e.g., [4], [5]. Competition models à la Cournot for the spot electricity market come in two general flavors: (i) prices due to transmission congestion over the network are exogenously determined by the system operator, and (ii) prices are determined endogenously through a nodal inverse demand function. Early works by Jing-Yuan et al. [7], Oren [8], and Willems [9] belong to the first category. Papers by Metzler et al. [10], and others [11]–[13] take the second approach. For a nuanced discussion of various Cournot competition models in electricity markets, we refer the reader to [10]. Perhaps the most related to the current work is [6], which studies the role of a market maker in networked Cournot competition.

II. MODEL AND PRELIMINARIES

We focus our study on a particular networked marketplace – electricity markets. We model the electricity market as a game between generators and the market maker as follows.

A. Notation

Let $\mathbb{R}$ denote the set of real numbers, and $\mathbb{R}_+$ denote the set of non-negative real numbers. For any two vectors $u, v$ of the same size, we say $u \geq v$ if the vector $u - v$ is element-wise non-negative. Let $\mathbf{1}$ denote the vector of all ones of appropriate size. For any vector $v \in \mathbb{R}^N$, let $v^\top$ denote its transpose, and $v_{-i}$ denote the vector of all elements in $v$ except the $i$-th element, i.e., $(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_N)^\top$.

B. Network model

We consider a power network with $n$ nodes (labelled $1, \ldots, n$) and $\ell$ edges. The line flows are related to the nodal power injections through Kirchhoff’s laws. While the true power flow model is nonlinear and defines a nonconvex feasible set, market operations are often studied with an idealized linear DC power flow model. We represent these constraints succinctly by an injection region $\mathcal{X}$, where:

$$\mathcal{X} := \{ x \in \mathbb{R}^N \mid -f \leq Hx \leq +f, \mathbf{1}^\top x = 0 \}. \quad (1)$$

The matrix $H_{\ell \times n}$ is known as the shift-factor matrix that depends on the admittances of the transmission lines of the power network. The transmission capacities of the lines are given by $f \in \mathbb{R}_+^\ell$. We refer the reader to [21], [22] for a detailed survey of this network model.

C. Market participants

Electricity markets have three sets of participants: (i) generators, (ii) consumers (represented in aggregate by the load serving companies), and (iii) the market maker (RTO/ISO). Put succinctly, we model generators as strategic agents and consumers as price-takers, and we model the market maker as a social planner that seeks to maximize social welfare while maintaining demand/supply balance throughout the network subject to the flow constraints.

Generators: At each node $k$ in the network, there is a generator $G_k$ that submits a quantity offer $q_k \geq 0$ and incurs a cost $c_k(q_k)$ for producing $q_k$. The map $c_k : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is assumed to be continuously differentiable, strictly increasing, and convex with $c_k(0) = 0$.

Consumers: We model price-taking consumers at each node $k$ with an aggregate inverse demand function $p_k : \mathbb{R}_+ \rightarrow \mathbb{R}$ that specifies how much the aggregate consumer is willing to pay to consume $d_k$ units of power. The function $p_k$ is assumed to be twice continuously differentiable, strictly decreasing, and concave, i.e., $p''_k < 0$ and $p'_k \leq 0$. Our model includes linear demand functions of the form $a_k - b_k d_k$, which are widely used in the literature.

Market maker: The system operator or the market maker $M$ chooses the $n \times 1$ vector $r := (r_1, \ldots, r_n)$ of rebalancing quantities, where $-r_k$ denotes the net power injection into node $k$. The market maker decides on the rebalancing quantities according to a market mechanism that allocates demands and prices across the network. The market mechanism balances the demand and supply through feasible flows over the network.

D. Payoff functions

Key to the model are the payoffs of the players in the market. Note that demand-supply balance at each node $k$ implies:

$$d_k := q_k + r_k,$$

which sets the price to $p_k(d_k) = p_k(q_k + r_k)$.

This models a nodal pricing mechanism that emulates the modus operandi in current electricity markets. In essence, the games we study have two sets of players: (i) generators $G_1, \ldots, G_n$, and (ii) the market maker $M$. The generator $G_k$ chooses $q_k$ while the market maker chooses $r$. The generator is paid at a rate given by the local nodal price $p_k(q_k + r_k)$. This in turn defines the profit (and hence the payoff) for generator $G_k$ as:

$$\pi_{G_k}(q_k, q_{-k}, r) := q_k p_k(q_k + r_k) - c_k(q_k). \quad (2)$$
The market maker $M$ is a social planner. Naturally, the market maker’s objective is the social welfare of the electricity market, defined as:

$$\pi_M(q, r) := \sum_{k=1}^{n} \left( \int_{0}^{q_k + r_k} \left( p_k(w_k)dw_k - c_k(q_k) \right) \right). \quad (3)$$

E. Strategy sets

The action of each player affects the set of strategies available to another player. In games with such coupled constraints, it is customary to define a set of constraints $S$, from which the strategy set of each player is derived. In particular, define:

$$S := \{(q, r) \in \mathbb{R}^n \times \mathbb{R}^n \mid q \geq 0, r \geq 0, r \in \mathcal{X}\}. \quad (4)$$

The strategy set of the market maker is given by:

$$\mathcal{S}_M(q) := \{r \in \mathbb{R}^n \mid (q, r) \in S\}.$$

We study two different classes of games with slightly different strategy sets to capture anticipatory behavior and non-anticipatory behavior.

$$\mathcal{S}_{G_k}(q_k, r) := \{q_k \geq 0 \mid ((q_k, q_{-k}), r) \in S\},$$

$$\mathcal{S}_{G_k}(q_{-k}) := \{q_k \geq 0 \mid ((q_k, q_{-k}), r) \in S\}.$$  

An implicit assumption in the above strategy sets is that generators have infinite supply capacities. Though such supply constraints clearly affect the equilibrium behavior, the unconstrained model and its analysis provide a starting point to derive valuable insights. Additionally, these assumptions are widely used in analyzing markets. For example, see [5], [20].

F. Competition models

So far we have delineated the players in the market and their actions, strategy sets and payoffs. Now, we define two different models of competition.2

1) Cournot competition: The generators $G_1, \ldots, G_n$, and the market maker $M$ engage in a simultaneous-move game.

2) Stackelberg competition: The generators participate in a simultaneous game at the first stage and the market maker $M$ follows by making a sequential-move.

The two formulations presented here differ based on who moves first. The difference between the models should be interpreted in terms of anticipatory behavior – or a lack thereof – on the part of the generators. In particular, consider the profit function $\pi_{G_k}$ as defined in (2). In a Stackelberg game, if $G_k$ tailors its quantity offer without anticipating the market maker’s response, this model reduces to the Cournot game.

G. Equilibrium concepts

The study of market outcomes requires a relevant equilibrium concept for the games in question. The strategy sets of the players being coupled, the generalized Nash equilibrium framework, developed by Arrow and Debreu [23], is a natural choice. The equilibrium concepts differ slightly between the Cournot and the Stackelberg models.

Formally, we say that an action profile $(q^*, r^*)$ constitutes a Cournot equilibrium, if:

$$\pi_{G_k}(q^*_k, q^-_{-k}, r^*) \geq \pi_{G_k}(q_k, q^-_{-k}, r^*),$$

for all $q_k \in S_{G_k}(q^*_k, r^*)$, $k = 1, \ldots, n$, and:

$$\pi_M(q^*_c, r^*) \geq \pi_M(q^*, r) \text{ for all } r \in \mathcal{S}_M(q^*).$$

To define a Stackelberg equilibrium, we introduce the following additional notation. Let $\mathcal{S}^r := \{q \mid (q, r) \in S\}$ denote the projection of $S$ to the coordinates in $r$. Define the reaction function that maps $q \in \mathcal{S}^r$ to $\rho(q) \in \mathcal{S}(q)$. An action profile $q^*$ and a map $\rho^*(\cdot)$ constitute a Stackelberg equilibrium, if:

$$\pi_{G_k}(q^*_k, q^-_{-k}, \rho^*(q^*_k, q^-_{-k})) \geq \pi_{G_k}(q_k, q^-_{-k}, \rho^*(q_k, q^-_{-k})),$$

for all $q_k \in S_{G_k}(q^*_k, q^-_{-k})$.

III. Summary of results

The focus of this paper is on understanding the impact of anticipatory behavior in networked competition. Within the context of our formulation, we thus contrast the equilibrium outcomes of the networked Cournot and networked Stackelberg models.

Our analysis focuses on both equilibrium existence and efficiency. In order to characterize the efficiency of equilibria outcomes, we adopt the notion of the price of anarchy (PoA), which is the worst-case ratio of the social welfare at an equilibrium outcome to the optimal achievable social welfare.

A. Existence and efficiency of Cournot equilibria

To provide context for the results characterizing equilibria in the networked Stackelberg model, we start by summarizing, and extending, results in the networked Cournot model. In particular, the following result characterizes the existence and efficiency of equilibria in networked Cournot competition.

**Proposition 1.** Suppose the Cournot game is feasible.  
1) An equilibrium always exists.  
2) The price of anarchy can be arbitrarily large, even in an unconstrained two-node network with affine inverse demand functions and quadratic cost functions.

Note that the existence of equilibria was proven in [6]; however the price of anarchy result is novel to this paper (see Section IV for a proof). It is perhaps surprising how bad the efficiency can be given that the market maker is attempting to

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2In real markets, the cost functions, the demand functions, and the network parameters like network topology, line admittances, and line capacities may not be common knowledge to all participants. We make the simplifying assumption of complete information, akin to previous examples in the literature. For example, see [5], [20].
clear the market by maximizing the social welfare. Moreover, the unbounded price of anarchy can be shown using the simplest non-trivial setting, i.e., in two-node networks with affine inverse demand functions.

B. Existence and efficiency of Stackelberg equilibria

Since Stackelberg competition considers anticipatory behavior on top of the networked Cournot model, which is already difficult to analyze, one can expect that it is challenging to obtain general results characterizing networked Stackelberg equilibria. Further, one might expect equilibria to both exist less often and be more inefficient (have a larger price of anarchy) than the networked Cournot model. Interestingly, however, there is a large regime where networked Stackelberg is actually simpler to analyze and more efficient than the networked Cournot model.

Perhaps surprisingly, if the network is unconstrained, it turns out that a general class of networked Stackelberg models (when the price intercepts of the inverse demand function are spatially homogenous) can be reduced to a classical (non-networked) Cournot competition (see Section V). This implies that the existence and price of anarchy results follow from known results on that (simpler) model. In contrast, such a reduction is not possible for an unconstrained networked Cournot model. As a result, the anticipatory nature of the networked Stackelberg model actually simplifies the analysis. Moreover, in such settings, the networked Stackelberg model has a smaller price of anarchy than the networked Cournot model – 3/2 instead of unbounded. In essence, the anticipatory behavior makes the marketplace more efficient!

It is, however, important to point out that, outside of the settings where such a reduction is possible, networked Stackelberg competition may lead to market failure, i.e., it may not admit an equilibrium.

Theorem 1 (Unconstrained network). Let \( f = \infty \) and suppose the Stackelberg game is feasible.

1) An equilibrium does not always exist. An equilibrium, however, always exists if the price intercepts of the inverse demand functions are spatially homogeneous.\(^3\)

2) The price of anarchy is bounded above by 3/2 when the inverse demand functions are affine with spatially homogenous price intercepts.

While the setting where we attain positive results may seem limited, note that the class of affine inverse demand functions is commonly studied in economics [24] and electricity markets [11]–[13]. An affine inverse demand function corresponds to a budget constrained aggregate consumer maximizing a quadratic utility function.

Not surprisingly, when network constraints are included, the characterization of networked Stackelberg equilibria becomes much more difficult. In such situations, it is challenging to provide general results about either existence or efficiency. In the case of existence, we can construct examples where no equilibrium exists even in the setting where the price intercepts of the inverse demand function are spatially homogenous. In the case of efficiency, obtaining bounds in two-node networks is already difficult. However, as the following theorem highlights, two-node networks, where the price intercepts of the inverse demand functions are spatially homogenous, have a similarly strong bound on the price of anarchy. This is reminiscent of the case with unconstrained networks.

Theorem 2 (Constrained network). Let \( f \) be finite and suppose the Stackelberg game is feasible.

1) An equilibrium does not always exist, even if the price intercepts of the inverse demand functions are spatially homogeneous.

2) The price of anarchy for 2-node networks is bounded above by 4/3, when the inverse demand functions are affine with spatially homogenous price intercepts and costs are quadratic.

Note that there is an additional assumption on the cost functions in the above price of anarchy result as compared with Theorem 1. While this may seem limiting, quadratic cost functions have been widely adopted in recent works that apply Cournot models to study electricity markets [6], [9], [10].

Further, while it may seem strange that Theorem 2 has a tighter bound on the price of anarchy than Theorem 1, note that when the line capacity is unconstrained, the same upper bound of 4/3 can be derived in the context 2-node networks with quadratic costs. This highlights the possibly surprising fact that adding network constraints does not reduce efficiency in this context.

Theorem 2 is far from a complete characterization of equilibria in the constrained network setting. However, it represents a provocative start. An exact characterization of existence and a more general characterization of efficiency in the constrained network setting are certainly interesting and challenging directions for future work.

IV. THE INEFFECTIVENESS OF NETWORKED COURNOT

In this section, we provide an example illustrating that the networked Cournot model has an unbounded price of anarchy, even in simple settings. This, combined with the existence results in [6], proves Proposition 1.

Consider a two-node network \((n = 2)\) where the transmission line joining nodes 1 and 2 has an infinite capacity, i.e., \( f = \infty \). Consider linear inverse demand functions of the form:

\[ p_1(d_1) = 1 - d_1, \quad p_2(d_2) = 1 - \beta d_2, \]

and quadratic cost functions of the form:

\[ c_1(q_1) = \gamma q_1^2, \quad c_2(q_2) = q_2^2. \]

\(^3\)That is, \( p_j(0) = p_k(0) \), for all \( 1 \leq j, k \leq n \).
For convenience, let \( r = r_1 = -r_2 \). Let \( (q_1^*, q_2^*, r^*) \) denote the social optimal allocation. Setting \( \partial \pi_M / \partial q_1, \partial \pi_M / \partial q_2, \partial \pi_M / \partial r \) to zero yields:

\[
\begin{align*}
q_1^*(2 + 2\gamma) + r^* &= 1, \\
q_2^*(2 + 2\gamma) - \beta r^* &= 1, \\
q_1^* - \beta q_2^* + (1 + \beta)r^* &= 0.
\end{align*}
\]

The last equation guarantees that \( r^* \in [-q_1^*, +q_2^*] \). Solving the above system and computing the social welfare, we get:

\[
\pi_M(q_1^*, q_2^*, r^*) = \frac{\beta \gamma + \beta + \gamma + 1}{6\beta + 2\beta + 4\gamma}.
\]

Let \( (q_1^*, q_2^*, r^*) \) define a Cournot equilibrium. Setting \( \partial \pi_{G_k} / \partial q_1, \partial \pi_{G_k} / \partial q_2 \), and \( \partial \pi_M / \partial r \) to zero gives:

\[
\begin{align*}
q_1^*(2 + 2\gamma) + r^* &= 1, \\
q_2^*(2 + 2\gamma) - \beta r^* &= 1, \\
q_1^* - \beta q_2^* + (1 + \beta)r^* &= 0,
\end{align*}
\]

Again, the last equation guarantees that \( r^c \in [-q_1^*, +q_2^*] \). One can show that the solution to the above system of linear equations defines a unique equilibrium in the Cournot game. The social welfare at this equilibrium is given in (5).

Substituting the expressions for social welfare into the price of anarchy \( \text{PoA} \), we have that:

\[
\lim_{\beta \to \infty} \text{PoA}(\beta, \gamma) = \lim_{\beta \to \infty} \pi_M(q_1^*, q_2^*, r^*) = \frac{4(2\gamma + 1)^3}{(2\gamma + 3)(3\gamma + 1)}.
\]

As \( \gamma \to \infty \), the above expression grows arbitrarily large, which shows that the price of anarchy can be unbounded.

V. STACKELBERG COMPETITION IN UNCONSTRAINED NETWORKS

In this section we focus on Stackelberg competition in unconstrained networks, and prove the existence and efficiency results in Theorem 1.

A. Existence

Crucial to our analysis of existence is the insight that, in the case where price intercepts are spatially homogenous, any equilibrium of our Stackelberg game is also an equilibrium of a classical non-networked Cournot game with an inverse demand curve aggregated from the individual nodal demand curves. This insight allows us to leverage existence results for classical Cournot games.

Let the common price intercept of the inverse demand functions be given by \( p^0 > 0 \). The inverse demand function \( p_k : \mathbb{R}_+ \to \mathbb{R} \) is assumed to be concave and monotonically decreasing, and hence \( p_k^{-1} : (-\infty, p^0) \to \mathbb{R}_+ \) is well-defined for each \( k = 1, \ldots, n \). Also, the domains of \( p_k^{-1} \) are identical for all \( k \), and hence the following map \( D : (-\infty, p^0) \to \mathbb{R}_+ \) is well-defined:

\[
D(x) := \sum_{k=1}^{n} p_k^{-1}(x).
\]

Since \( p_k(\cdot) \) for \( k = 1, \ldots, n \) is monotonically decreasing, it follows that \( D(\cdot) \) is monotonically decreasing. Furthermore, its inverse \( D^{-1} : [0, \infty) \to (-\infty, p^0) \) exists. For convenience, we will refer to \( D^{-1} \) as the aggregate inverse demand function. The following result establishes the relationship between the networked Stackelberg game and a classical Cournot game.

Lemma 1. Let \((q^*, \rho^*(\cdot))\) be an equilibrium of the networked Stackelberg game. Then \( q^* \) is an equilibrium of a corresponding classical non-networked Cournot game between the \( n \) generators with a market demand function \( D(\cdot) \). Hence, each generator \( G_k \) in the corresponding classical non-networked Cournot game has a payoff, given by:

\[
D^{-1}(q_k + 1^T q_{-k}) : q_k - c_k(q_k).
\]

Proof: First, we compute a candidate equilibrium reaction function \( \rho^* \). For any \( q \in S^* \), let \( d^*(q) := q + \rho^*(q) \). The definition of \( \rho^* \) implies that \( d^*(q) \) is a solution to the following optimization problem:

\[
\begin{align*}
\text{minimize} & \quad \sum_{k=1}^{n} \left[ \int_{0}^{d_k} p_k(w_k) dw_k - c_k(q_k) \right], \\
\text{subject to} & \quad 1^T d = 1^T q.
\end{align*}
\]

The objective function in (7) is strictly concave, and hence there is a unique solution to the above optimization problem. That in turn defines a unique equilibrium reaction function. Associate a Lagrange multiplier \( \lambda \in \mathbb{R} \) to the equality constraint in (7). The Lagrangian function is then given by:

\[
\sum_{k=1}^{n} \left[ \int_{0}^{d_k} p_k(w_k) dw_k - c_k(q_k) \right] - \lambda \left( 1^T d - 1^T q \right).
\]

Differentiating the Lagrangian and setting it to zero, we get \( p_k(d_k^*(q_k)) = \lambda^* \), where \( \lambda^* \) is the Lagrange multiplier at optimality. Utilizing the equality constraint in (7), we obtain \( \lambda^* = D^{-1}(1^T q) \), and hence

\[
d_k^*(q) = p_k^{-1} \left( D^{-1}(1^T q) \right).
\]

From the properties of \( p_k, d^*(q) \) is component-wise positive and monotonically increasing in each coordinate \( q_k \). The equilibrium reaction function is then given by \( \rho^*(q) = d^*(q) - q \). Substituting into the payoff of generator \( G_k \) gives:

\[
\pi_k(q_k, q_{-k}, \rho^*(q_k, q_{-k})) = p_k(d_k^*(q_k)) q_k - c_k(q_k)
\]

\[
= D^{-1}(q_k + 1^T q_{-k}) q_k - c_k(q_k).
\]
This expression is reminiscent of a producer with a cost function $c_k$ facing an inverse demand function $D^{-1}$ in a classical non-networked Cournot game. For an analysis of this classical model, we refer the reader to [17], [25].

We are only left to show that $π_k(q_k, q_{−k}, p^∗(q_k, q_{−k}))$ is concave in $q_k ≥ 0$. To see this, observe that a concave and strictly decreasing function has an inverse with the same properties. Thus, $p_k^{-1}$, $k = 1, . . . , n$, is concave and strictly decreasing, which in turn implies that $D$ and its inverse are also concave and strictly decreasing. The rest follows from a well known result, that an equilibrium exists in a classical non-networked Cournot game. For an analysis of this game could be arbitrarily high under general concave demand functions. However, when the inverse demand functions (at nodes $k = 1, 2$) are given by $p_k(d_k) = 1 - d_k$, and $c_k(q_k) = c_kq_k^2$, respectively and $c_1 > c_2 > 0$. In particular, we can show that if the capacity of the line $f$ joining buses 1 and 2 satisfies:

$$f ≥ \frac{c_1 - c_2}{4(c_1 + 1)(c_2 + 1) - (c_1 + 1) - (c_2 + 1)},$$

$$f ≤ \frac{c_1 - c_2}{(1 + 2c_1)(1 + 2c_2) - 1/4},$$

an equilibrium does not exist in the Stackelberg game. See the Appendix for details.

B. Efficiency

We now move to characterizing the price of anarchy of Stackelberg equilibria in unconstrained networks. These results follow quickly from the observation in Lemma 1 that any equilibrium of our Stackelberg game is also an equilibrium of a corresponding classical Cournot (Nash) game with a concave inverse demand function (expressed in Eq. (6)).

The only additional observation necessary is that the socially optimal allocation in the Stackelberg game is the same as the socially optimal allocation in the corresponding classical Cournot game.

The above two facts imply that existing results on the price of anarchy of classical Cournot games apply directly to our networked Stackelberg game. For instance, the result in [19] implies that the price of anarchy of our Stackelberg game could be arbitrarily high under general concave demand functions. However, when the inverse demand functions (at all nodes) are affine with homogenous price intercepts (which implies that the aggregate demand function in (6) is affine), the result in [18, Theorem 12] implies that the price of anarchy of our Stackelberg game has an upper bound of 3/2.

VI. STACKELBERG COMPETITION IN CONSTRAINED NETWORKS

As mentioned in Section III, it is natural to expect Stackelberg equilibria to be much more difficult to characterize in the context of constrained networks. This can be seen in the generality of the results we obtain and the fact that the analysis needed to obtain the (less general) results is more involved.

A. Existence

Existence is far from guaranteed in this setting. In fact, even if the price intercepts of the demand functions are uniform across the network, an equilibrium may not exist when there is some line with finite capacity. To highlight this, we provide an explicit example with a two-node network ($n = 2$) where the demand and cost functions at nodes $k = 1, 2$ are given by $p_k(d_k) = 1 - d_k$, and $c_k(q_k) = c_kq_k^2$, respectively and $c_1 > c_2 > 0$. In particular, we can show that if the capacity of the line $f$ joining buses 1 and 2 satisfies:

$$f ≥ \frac{c_1 - c_2}{4(c_1 + 1)(c_2 + 1) - (c_1 + 1) - (c_2 + 1)},$$

$$f ≤ \frac{c_1 - c_2}{(1 + 2c_1)(1 + 2c_2) - 1/4},$$

an equilibrium does not exist in the Stackelberg game. See the Appendix for details.

B. Efficiency

In contrast with the negative result on existence, we can prove a strong positive result about efficiency, albeit in the limited setting of a two-node network. Recall that we have established an upper bound of 3/2 in Theorem 1 for the price of anarchy of a Stackelberg game in unconstrained networks.

The following lemma establishes a refined price of anarchy bound in the setting of Lemma 2, but with finite line capacity 4 omitted due to space constraints. We seek to establish the price of anarchy of a Stackelberg game in unconstrained networks. Consider a network with $n$ nodes where the line capacity were $f = ∞$, then $h(q_1, q_2)$ would define the market maker’s best response to the generator’s actions $(q_1, q_2)$. If the line capacity were finite, then $h(q_1, q_2)$ would define the market maker’s best response to the generator’s actions $(q_1, q_2)$.

To characterize the equilibrium behavior, consider the equilibrium actions of the generator $q^∗ = (q_1^∗, q_2^∗)$. Then, $h(q^∗)$ must satisfy one of following three cases: (i) $h(q^∗) = f$, (ii) $h(q^∗) > f$ or $h(q^∗) ∈ (−f, f)$. Each case is tackled

4The model being symmetric with the nodes, $h(q^∗) < −f$ is equivalent to the case with $h(q^∗) > f$. 
(i) The Stackelberg equilibrium can be characterized by three different first-order conditions, corresponding to the profit-maximization of the generators and the social welfare maximization by the market maker. One can show that these conditions cannot be simultaneously satisfied when \( h(q^*) = f \), implying an equilibrium of this form does not exist.

(ii) For an equilibrium with \( h(q^*) > f \), it can be argued that \( \rho^*(q^*) = f \). Further, if \( q^* \) denotes the socially optimal productions, it can be shown that \( h(q^*) > f \) and \( r^* = f \). In essence, the congestion of the line in equilibrium implies a congestion in the socially optimal outcome. In summary, we have \( h(q^*) > f, \rho^*(q^*) = f \implies h(q^*) > f, r^* = f \).

Leveraging the first order conditions, it can be further shown that the analysis of the price of anarchy of the entire market reduces to one of studying two separate markets, where the outcomes (in equilibrium and at social optimum) at each node depends only on the local demand functions and the local generation cost at that node. Obtaining the PoA bound then relies on existing results for monopoly markets; specifically [19, Theorem 3].

(iii) Finally, suppose \( h(q^*) \in (-f, f) \). If \( r^* < f \), then the rest follows from Lemma 2. Otherwise, the optimal social welfare at \( (q_1^*, q_2^*, r^* = f) \) is bounded above by the optimal social welfare obtained with \( f = \infty \). Again, Lemma 2 provides the necessary bound.

REFERENCES


APPENDIX

A. Examples with no Stackelberg equilibrium

In this section, we provide examples where there is no Stackelberg equilibrium when price intercepts are not spatially homogeneous or when the network is constrained.

1) An unconstrained network: Consider a two-node unconstrained network, i.e., \( n = 2 \) and \( f = \infty \), where the demand and cost functions are given by \( p_k(d_k) = a_k - d_k \) and \( c_k(q_k) = q_k^2 \) respectively for \( k = 1, 2 \). Suppose \( a_1 > a_2 > 0 \). For convenience, let \( r = r_1 = -r_2 \). Then, the payoff of the market-maker is given by:

\[
\pi_M(q_1, q_2, r) = a_1(q_1 + r) - \frac{1}{2}(q_1 + r)^2 + a_2(q_2 - r) - \frac{1}{2}(q_2 - r)^2.
\]

Maximizing \( \pi_M \) over \( r \in [-q_1, +q_2] \), we obtain the optimal reaction function:

\[
\rho^*(q_1, q_2) = \begin{cases} 
q_2, & \text{if } q_1 + q_2 \leq a_1 - a_2; \\
(a_1 - a_2 - q_1 + q_2)/2, & \text{otherwise.}
\end{cases}
\]

Suppose \( (q_1^*, q_2^*, \rho^*) \) constitute a Stackelberg equilibrium. We consider three cases separately, where \( q_1^* + q_2^* \) is greater, less, or equal to \( a_1 - a_2 \).
Let the Stackelberg equilibrium does not exist if \( c \) which in turn yields 
\[
q_i^* = \frac{(a_1 + a_2)}{8}, \quad \text{and} \quad q_j^* = \frac{a_2}{2}.
\]
Substituting the above relations in \( q_i^* + q_j^* < a_1 - a_2 \), we obtain \( a_1/a_2 > (11/6) \).

Case III: \( q_i^* + q_j^* = a_1 - a_2 \). We show that there does not exist such an equilibrium. Substituting \( \rho^*(q_i, q_j) \) in \( G_1 \)’s payoff function for \( q_i = (a_1 - a_2 - q_j^*) \pm \varepsilon \), the optimality of \( q_i^* \) yields 
\[
a_1 - 4q_i^* - q_j^* \geq 0, \\
a_1 - 4q_i^* - q_j^* + (1/2)q_j^* \leq 0.
\]
Utilizing \( q_i^* = a_1 - a_2 - q_j^* \) in the above inequalities then imply:
\[
a_1 - (4/3)a_2 \leq q_j^* \leq a_1 - (7/5)a_2,
\]
which in turn yields \( 4/3 < 7/5 \). This is a contradiction and hence case III cannot arise.

From the analysis of cases I and II, we conclude that Stackelberg equilibrium does not exist if \( a_1/a_2 \in (9/5, 11/6) \).

2) A constrained network: Consider a two-node network (\( n = 2 \)). Let the demand and cost functions at node \( k = 1, 2 \) be given by \( D_k(d_k) = 1 - d_k \), and \( c_k(q_k) = c_k q_k^2 \), where \( c_1 > c_2 > 0 \). For convenience, let \( r = r_1 = -r_2 \). The reaction function at equilibrium can be shown to satisfy:
\[
\rho^*(q_i, q_j) = \begin{cases} 
-f & \text{if } q_i > q_j + 2f, \\
f & \text{if } q_i < q_j - 2f, \\
(1/2)(-q_i + q_j) & \text{otherwise}.
\end{cases}
\]
Let \( (q_i^*, q_j^*, \rho^*) \) denote a Stackelberg equilibrium. The following five possible cases arise for the equilibrium quantities: (I) \( q_i^* < q_j^* - 2f \), (II) \( q_i^* = q_j^* - 2f \), (III) \( q_i^* > q_j^* - 2f \), and \( q_i^* < q_j^* + 2f \), (IV) \( q_i^* = q_j^* + 2f \), and (V) \( q_i^* > q_j^* + 2f \). Each case is handled separately.

Case I: \( q_i^* < q_j^* - 2f \). The first-order conditions for \( G_1 \) and \( G_2 \) at \( q_i^* \) and \( q_j^* \) are given by:
\[
1 - 2q_i^* - f - 2c_1q_j^* = 0, \\
1 - 2q_j^* + f - 2c_2q_j^* = 0,
\]
which in turn yields
\[
q_i^* = \frac{1 - f}{2(c_1 + 1)}, \quad q_j^* = \frac{1 + f}{2(c_2 + 1)}.
\]
Utilizing \( q_i^* < q_j^* - 2f \), we obtain
\[
f < \frac{c_1 - c_2}{4(c_1 + 1)(c_2 + 1) - (c_1 + 1) - (c_2 + 1)}.
\]
The above contradicts the hypothesis in (8). Hence, this case cannot arise.

Case II: \( q_i^* = q_j^* - 2f \).

The first-order conditions for \( G_1 \) imply:
\[
1 - 2q_i^* - f - 2c_1q_i^* \geq 0; \\
1 - 2q_j^* - (1/2)(-q_i^* + q_j^*) + (1/2)q_j^* - 2c_1q_i^* \leq 0.
\]
Substituting \( q_j^* = q_i^* - 2f \), it can be checked that the above two equations cannot be simultaneously true. Thus, case II never arises.

Case III: \( q_i^* > q_j^* - 2f \), and \( q_i^* < q_j^* + 2f \). Again, the first-order conditions for \( G_1, G_2 \) are given by:
\[
1 - 2q_i^* - (1/2)(-q_i^* + q_j^*) + (1/2)q_j^* - 2c_1q_i^* = 0, \\
1 - 2q_j^* + (1/2)(-q_i^* + q_j^*) + (1/2)q_j^* - 2c_2q_j^* = 0.
\]
which in turn yields
\[
q_i^* = \frac{(1/2) + 2c_3}{(1 + 2c_1)(1 + 2c_2) - 1/4}, \\
q_j^* = \frac{(1/2) + 2c_1}{(1 + 2c_1)(1 + 2c_2) - 1/4}.
\]
Utilizing the above equalities in the hypothesis \( q_i^* > q_j^* - 2f \), and \( q_i^* < q_j^* + 2f \), we obtain
\[
f > \frac{c_1 - c_2}{(1 + 2c_1)(1 + 2c_2) - 1/4}.
\]
The above contradicts the hypothesis in (9). Hence, this case cannot arise.

Case IV: \( q_i^* = q_j^* + 2f \).

The first-order conditions for \( G_2 \) imply that:
\[
1 - 2q_i^* + f - 2c_2q_i^* \geq 0; \\
1 - 2q_j^* + (1/2)(-q_i^* + q_j^*) - (1/2)q_j^* - 2c_2q_j^* \leq 0.
\]
Substituting \( q_i^* = q_j^* + 2f \), it can be checked that the above two equations cannot be simultaneously true. Thus, case IV never arises.

Case V: \( q_i^* > q_j^* + 2f \). The first-order conditions for both generators imply that:
\[
q_i^* = \frac{1 + f}{2(c_1 + 1)}, \quad q_j^* = \frac{1 - f}{2(c_2 + 1)}.
\]
The above values cannot satisfy the hypothesis \( q_i^* > q_j^* + 2f \), and hence this case never arises.