Overcoming the Limitations of Utility Design for Multiagent Systems

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Abstract

Cooperative control focuses on deriving desirable collective behavior in multiagent systems through the design of local control algorithms. Game theory is beginning to emerge as a valuable set of tools for achieving this objective. A central component of this game theoretic approach is the assignment of utility functions to the individual agents. Here, the goal is to assign utility functions within an "admissible" design space such that the resulting game possesses desirable properties. Our first set of results illustrates the complexity associated with such a task. In particular, we prove that if we restrict the class of utility functions to be local, scalable, and budget-balanced then (i) ensuring that the resulting game possesses a pure Nash equilibrium requires computing a Shapley value, which can be computationally prohibitive for large-scale systems, and (ii) ensuring that the allocation which optimizes the system level objective is a pure Nash equilibrium is impossible. The last part of this paper demonstrates that both limitations can be overcome by introducing an underlying state space into the potential game structure.

I. Introduction

The central goal in cooperative control problems is to derive desirable collective behavior in multiagent systems through the design of local control algorithms [1]–[8]. There is an emerging interest in using game theory for this purpose [6], [9]–[14]. This design choice entails a two step process. The first step, which we refer to as *game design*, involves modeling the interactions of the agents in a game theoretic framework where the agents are designed as self-interested decision making entities. Specifically, this step involves defining the set of decision makers, their respective choices, and a local utility function for each agent used to guide behavior. The second step, which we refer to as *learning design*, involves specifying a distributed learning

This research was supported by AFOSR FA9550-09-1-0538 and NSF CCF-0830511

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algorithm that enables the agents to reach a desirable operating point, e.g., a Nash equilibrium of the designed game. The goal is to complete both design steps to ensure that the resulting distributed control algorithm meets the operational constraints while ensuring that the emergent behavior is desirable with respect to the system level objective.

The emerging interest in game theory for the design and control of multiagent systems stems from two key attributes of the approach. The first attribute is that game theory provides a framework for analyzing distributed systems where the decision making entities respond to heterogenous objectives and incomplete information. These decision makers could be self-interested, as is the case of social systems, or programmable components, as is the case in engineering systems. The second attribute is that game theory provides a natural decomposition between the distribution of the optimization problem (*game design*) and the specific local agent decision rules (*learning design*) by imposing restrictions on the structure of the underlying game [14]. For example, several recent papers focus on the framework of *potential games* [15] as a mediating layer for this decomposition [6], [8], [14], [16]–[19]. By designing the interaction framework as a potential game, the system designer can appeal to existing results from the theory of learning in games which provides several distributed learning algorithms with proven asymptotic guarantees [20]–[27].

The overarching goal of this research effort connecting game theory to cooperative control is to exploit this decomposition by developing a broad set of tools for both game design and learning design to meet the constraints and objectives for distributed control of multiagent systems.

This paper focuses on the game design component of this decomposition. In particular, the goal of this research effort is to identify the viability of potential games, or more generally strategic form games, as a mediating layer for this decomposition. We gauge the viability by identifying whether *systematic methodologies* can be developed for designing games with desirable properties. The two dominant properties that we focus on in this paper are (i) ensuring that the resulting game is a potential game (or more generally possesses a pure Nash equilibrium[1]) and (ii) ensuring that the resulting equilibria are efficient with respect to the system level objective. The efficiency guarantees, commonly referred to as the *price of anarchy* [28], represents a competitive ratio for the distributed algorithm that results from a designed game coupled with a distributed learning algorithm.

[1]We henceforth refer to a pure strategy Nash equilibrium as just an equilibrium.

Utility design for social systems has been studied extensively in the game theoretic literature, e.g., cost sharing problems [29]–[34] and mechanism design [35]; however the difference between the constraints and objectives pertaining to social and engineering systems requires looking at this literature from a new perspective. The focus in social systems is to ensure the *stability* of desirable outcomes by manipulating agent utility functions in an admissible fashion. On the other hand, the focus in engineering systems is to establish a dynamical process that converges to an efficient outcome. Here, utility functions are introduced "behind the scenes" to play the role of target functions for each agent's control policy.

Nonetheless, many of the methodologies developed for cost sharing problems have immediate applicability to utility design for engineering multiagent systems. Two examples are the *marginal contribution* and *(weighted) Shapley value*, which both guarantee that the resulting interaction framework represents a potential game irrespective of the structure of the resource allocation problem [2]. These rules will be described in more detail in Section II-D, but for now the key point is that the marginal contribution design is not budget-balanced[2] but maintains a price of stability[3] of 1, while the weighted Shapley value design is budget-balanced but does not maintain a price of stability of 1.

Thus, at first glance, it appears that the marginal contribution design outperforms the weighted Shapley value design as the importance of budget-balanced utility functions for engineering systems is unclear. However, several recent results have identified a connection between price of anarchy guarantees and budget-balanced (or more generally budget-constrained) utility functions in both networked resource allocation problems [36]–[38] and more traditional cost sharing problems [30], [32].[4] Furthermore, it turns out that the Shapley value utility is provably optimal for many specific problem settings, e.g., network coding [8] and network formation [37]. That is, the Shapley value utility optimizes the price of anarchy over a broad class of admissible protocols

[2]Utility functions are budget-balanced if the sum of the agent utility functions is equivalent to the system level objective.

[3]The (price of stability, price of anarchy) provides a worst case measure of the efficiency associated with (the best, any) equilibrium. A (price of stability, price of anarchy) of $x \in [0, 1]$ means that system level objective associate with (the best, any) equilibrium is at least within a multiplicative factor x of the optimal system level objective.

[4]Consider the results in [36] which provides a price of anarchy of $1/2$ for a class of networked resource allocation problems with submodular welfare functions provided that the sum of agents' utility functions is less than or equal to the total welfare. The results in [37] impose the same constraint on agents' utility functions. Furthermore, see the proof of Theorem 4.1 to see how budget-balanced protocols are exploited for providing price of anarchy guarantees.

including both budget-balanced and non-budget-balanced protocols.[5] While the existing literature does not provide a systematic utility design methodology for optimizing the price of anarchy, the connection between budget-balanced utility functions and price of anarchy guarantees is evident. Accordingly, the focus of this work is on identifying systematic utility design methodologies that yield budget-balanced utility functions and also ensure the existence of an equilibrium irrespective of the specific resource allocation problem.

The first two results in this paper are negative and highlight the computational complexity associated with meeting such an objective. Our first results provides a complete characterization of all such methodologies. In particular, we prove that *a utility design methodology yields budget-balanced utility functions and also ensures the existence of an equilibrium if and only if the methodology is equivalent to a weighted Shapley value.* Therefore, if a system designer requires the use of budget-balanced utility functions, either for social considerations or for providing desirable efficiency guarantees, weighted Shapley values represent the entire set of feasible design methodologies. Hence, this identifies the complete design space that a system planner would need to consider for meeting the above objectives. The limitation of utility design for engineering systems hinges on the fact that in many setting computing a weighted Shapley value is intractable. This result can be considered complimentary to the results in [39], which derives a similar result for a particular class of cost minimization problems.

A secondary issue associated with using budget-balanced utility functions is a degradation in the price of stability. Informally, our second result proves that *it is impossible to guarantee that the optimal allocation is an equilibrium when using budget-balanced utility functions.* This result highlights a fundamental tradeoff between the price of stability and budget-balanced utilities as non-budget-balanced utilities can always be designed that guarantee a price of stability of 1, e.g., the marginal contribution [2], [40]. The price of stability is an important efficiency measure as there are distributed learning algorithms that can frequently reach these best equilibria [41]–[43].

The third contribution of this work shifts from identifying limitations of utility design to overcoming these limitations by conditioning utility functions on additional information, i.e., a

[5] Consider the problem of network coding in [8]. The price of anarchy associated with the marginal contribution is 0, meaning that an equilibrium could be arbitrarily bad with respect to the system optimal. On the other hand, the price of anarchy associated with the Shapley value is $1/2$, meaning that any equilibrium will be at least 50% as good as the optimal allocation. Furthermore, the Shapley value optimizes the price of anarchy in this setting.

state variable. Specifically, we introduce a *dynamic ordered protocol* which makes use of a local state variable that overcomes both of the derived limitations by giving rise to a budget-balanced protocol that maintains a price of stability of 1 across all games. The key idea behind this new utility design is to change the underlying game structure so that it is a specific form of a stochastic game [44] termed state based game [45]. Our dynamic ordered protocol outperforms both the marginal contribution and the Shapley value in all attributes when restricting attention to resource allocation problems with submodular objective functions as highlighted in Table I.

Distribution Rule	NE Exists	Budget Balanced	Tractable	PoS	PoA
Marginal contribution	yes	no	yes	1	1/2
Shapley value	yes	yes	no	1/2	1/2
Dynamic ordered protocol	**yes**	**yes**	**yes**	**1**	**1/2**

TABLE I

SUMMARY OF PROTOCOLS FOR DISTRIBUTED WELFARE GAMES WITH SUBMODULAR WELFARE FUNCTIONS.

II. PRELIMINARIES

A. A Model for Resource Allocation

We consider a class of resource allocation problems where there exists a set of agents N and a finite set of resources $\mathcal{R}$ that are to be shared by the agents. Each agent $i \in N$ possesses an action set $\mathcal{A}_i \subseteq 2^{\mathcal{R}}$ where $2^{\mathcal{R}}$ denotes the power sets of $\mathcal{R}$; therefore, an agent may have the option of selecting multiple resources. An action profile, or allocation, is represented by an action tuple $a = (a_1, a_2, ..., a_n) \in \mathcal{A}$ where the set of action profiles is denoted by $\mathcal{A} := \mathcal{A}_1 \times \cdots \mathcal{A}_n$. We frequently express an allocation a as (a_i, a_{-i}) where $a_{-i} := (a_1, ..., a_{i-1}, a_{i+1}, ..., a_n)$ represents the action of all agents other than agent i in the allocation a. We restrict our attention to the class of *separable welfare functions*[6] of the form

$$W(a) = \sum_{r \in \mathcal{R}} W_r\left(\{a\}_r\right)$$

[6]Separable welfare functions can model the global objective for several important classes of resource allocation problems including routing over a network [28], vehicle target assignment problem [1], sensor coverage [2], [46], content distribution [47], graph coloring [48], network coding [8], among many others. Furthermore, the class of separable welfare function imposes a natural notion of "locality" for which we seek to satisfy in the design of agent utility functions.

where $W_r : 2^N \to R^+$ is the welfare function for resource r and $\{a\}_r := \{i \in N : r \in a_i\}$ is the set of agents using resource r in the allocation a. Note that when restricting attention to separable welfare functions, the welfare generated at a particular resource depends only on which agents are currently using that resource. In general, a systems designer would like to find an allocation that optimizes the system welfare

$$a^{\text{opt}} \in \arg\max_{a \in \mathcal{A}} W(a).$$

Example #1: Coverage Problems: Consider a coverage problem consisting of a finite set of resources (or targets) denoted by $\mathcal{R}$ where each resource $r \in \mathcal{R}$ has a relative value $v_r \geq 0$, e.g., the vehicle target assignment problem [1]. Each agent $i \in N$ is parameterized with a success probability denoted by $p_i(r, a_i) \in [0, 1]$ which indicates the probability that agent i will successfully "cover" resource r given the assignment a_i. The goal of a coverage problem is to find a joint assignment that maximizes the global welfare function

$$W(a) = \sum_{r \in \mathcal{R}} v_r \left(1 - \prod_{i \in \{a\}_r} [1 - p_i(r, a_i)] \right) \tag{1}$$

where $\left(1 - \prod_{i \in \{a\}_r} [1 - p_i(r, a_i)]\right)$ represents the probability that resource r is successfully covered by the joint assignment a. The goal in this setting is to establish local control policies for the individual agents to ensure convergence to a desirable allocation. Furthermore, in many problem settings the control design is further complicated since the number of targets or the topological structure of the actions sets $\{\mathcal{A}_i\}_{i \in N}$ is not known a priori.[7]

Example #2: Routing Problems: A routing problem consists of a group of agents that seek to utilize a common network characterized by a vertex set V and an edge set $E \subseteq V \times V$. Each agent possesses an action set $\mathcal{A}_i \subseteq 2^E$ where each action choice $a_i \in \mathcal{A}_i$ represents a collection of edges that satisfies individual demands, e.g., connecting a source and destination. There is an underlying cost associated with each edge in the network which depends on the subset of agents

[7]We will frequently restrict our attention to *submodular* welfare functions which are a common feature of many objective functions for engineering applications ranging from content distribution [47] to coverage problems [1], [49]. A welfare function W_r is submodular if for any agent set $S \subseteq T \subseteq N$ and any agent $i \in N$ we have $W_r(S \cup \{i\}) - W_r(S) \geq W_r(T \cup \{i\}) - W_r(T)$. Note that (1) is submodular.

using that edge, i.e., given the allocation a the cost of edge $e \in E$ is $c_e(\{a\}_e)$.[8] The goal is to establish local control policies for the individual agents to ensure convergence to an allocation that minimizes an aggregate network cost of the form

$$C(a) = \sum_{e \in E} c_e(\{a\}_e). \tag{2}$$

In many settings this control design is further complicated since the topology of the network, i.e., the topological structure of the actions sets $\{\mathcal{A}_i\}_{i \in N}$, is typically not characterized.

B. Admissible Utility Design Space

We consider a game theoretic model for the above resource allocation problems. Here, we focus on the design of admissible utility functions $\{U_i\}_{i \in N}$ for the agents which meets two fundamental requirements. The first requirement, which we refer to as *locality*, is that an agent's utility function should only depend on local information. More specifically, the utility of agent i given the allocation a should only depend on (i) the resources $r \in a_i$ and (ii) the other agents that used resources $r \in a_i$.[9] Accordingly, we focus on the design of utility functions of the following form:

Definition 2.1 (Locality): The utility function for agent $i \in N$ is *local* if for any allocation $a \in \mathcal{A}$ the utility is of the form

$$U_i(a_i, a_{-i}) = \sum_{r \in a_i} f_r(i, \{a\}_r) \tag{3}$$

where $f_r : N \times 2^N \to R$ is referred to as the *protocol* at resource r.

Note that a fixed set of protocols $\{f_r\}_{r \in \mathcal{R}}$ results in a well defined game for any collection of action sets $\{\mathcal{A}_i\}_{i \in N}$. We define such a game, termed *distributed welfare game* [2], by the tuple $G = \{N, \mathcal{R}, \{\mathcal{A}_i\}, \{W_r\}, \{f_r\}\}$ where we omit the subscripts on the sets $\{\cdot\}$ for brevity. We refer to the set of distributed welfare games as all games of this form.

[8] In the classical congestion games [15], [28], the cost of an edge is the total congestion on that edge which is of the form $c_e(\{a\}_e) = |a|_e \cdot \tilde{c}_e(|a|_e)$ where $|a|_e$ is the cardinality of the set $\{a\}_e$ and $\tilde{c}_e(|a|_e)$ represents the congestion on link e which only depends on the number of agents using that edge. In network coding with "reverse carpooling" over a wireless network [8], the cost function on a particular edge $(ij) \in E$ represents the number of transmissions necessary to facilitate the exchange of information and is of the form $c_{ij}(\{a\}_{ij}) = \max\{|a|_{i \to j}, |a|_{j \to i}\}$ where $|a|_{i \to j}$ represents the number of agents using the directed edge $i \to j$ given the allocation a.

[9] For alternative classes of non-separable resource allocation problems the appropriate measure of locality would need to be specified.

Our second requirement, which we refer to as *scalability*, imposes the constraint that utility functions must be defined without relying on specific information regarding the topological structure of the action sets $\{\mathcal{A}_i\}_{i \in N}$. Our motivation for this constraints is that such information is not available to the system designer in many settings, e.g., routing problems or vehicle target assignment problems.

Definition 2.2 (Scalability): The set of protocols $\{f_r\}_{r \in \mathcal{R}}$ is scalable if for any two resources $r, r' \in \mathcal{R}$ we have

$$W_r(S) = W_{r'}(S) \ \forall \ S \subseteq N \ \Rightarrow \ f_r(S) = f_{r'}(S) \ \forall \ S \subseteq N. \tag{4}$$

C. Performance Metrics

The focus of this work is to identify mechanisms for attaining distributed control strategies that converge to an efficient allocation in the above resource allocation problems. Accordingly, for a given class of resource allocation problem parameterized by an agent set N, a resource set $\mathcal{R}$, and welfare functions $\{W_r\}$, we focus on the design of protocols $\{f_r\}$ that provides the following guarantees for all games G in the set $\mathcal{G} = \{N, \mathcal{R}, \{\mathcal{A}_i\}, \{W_r\}, \{f_r\}\}_{\mathcal{A}_i \subseteq 2^{\mathcal{R}}}$.[10]

Existence of equilibria: An equilibrium represents an allocation where no single agent has a unilateral incentive to deviate. More formally, an allocation a^* is an equilibrium if

$$U_i(a_i^*, a_{-i}^*) = \max_{a_i \in \mathcal{A}_i} U_i(a_i, a_{-i}^*).$$

Potential game: A game is a *weighted potential games* [15] if there exists a potential function $\phi : \mathcal{A} \to \mathbb{R}$ and a positive weight for each agent $w_i > 0$ such that for every agent $i \in N$, for every $a_{-i} \in \mathcal{A}_{-i}$, and for every $a_i', a_i'' \in \mathcal{A}_i$,

$$\phi(a_i', a_{-i}) - \phi(a_i'', a_{-i}) = w_i \left(U_i(a_i', a_{-i}) - U_i(a_i'', a_{-i}) \right). \tag{5}$$

When this condition is satisfied, the game is called a weighted potential game with the potential function ϕ. In the special case when $w_i = 1$ for all agents $i \in N$, the game is referred to as an exact potential game. It is easy to see that in potential games any action profile maximizing the

[10]Considering the set of games defined by extending action sets to all possible combinations is quite general and can accommodate both variations in the set of agents and variations in the number of resources. For example, setting $\mathcal{A}_i = \emptyset$ is equivalent to removing agent i from the game. Similarly, letting the action sets satisfy $\mathcal{A}_i \subseteq 2^{\mathcal{R} \setminus \{r_0\}}$ for each agent i is equivalent to removing resource r_0 from the specified resource allocation problem.

potential function is an equilibrium, hence every weighted potential game possesses at least one such equilibrium.

Efficiency of equilibria: We use the *price of anarchy (PoA)* and *price of stability (PoS)* to measure the efficiency of equilibria [33]. The price of anarchy gives a lower bound on the global welfare achieved by any equilibrium while the price of stability gives a lower bound on the global welfare associated with the best equilibrium. Specifically, for any particular game $G \in \mathcal{G}$ let $\mathcal{E}(G)$ denote the set of equilibria, $PoA(G)$ denote the price of anarchy, and $PoS(G)$ denote the price of stability for the game G where

$$PoA(G) := \min_{a^{\rm ne} \in \mathcal{E}(G)} \frac{W(a^{\rm ne})}{W(a^{\rm opt})} \tag{6}$$

$$PoS(G) := \max_{a^{\rm ne} \in \mathcal{E}(G)} \frac{W(a^{\rm ne})}{W(a^{\rm opt})}, \tag{7}$$

where $a^{\rm opt} \in \arg\max_{a^* \in \mathcal{A}} W(a^*)$. The price of anarchy and price of stability for the set of games $\mathcal{G}$ is defined as $PoA(\mathcal{G}) := \inf_{G \in \mathcal{G}} PoA(G)$ and $PoS(\mathcal{G}) := \inf_{G \in \mathcal{G}} PoS(G)$ respectively.

D. Examples of Protocols

The posed utility design question represents a cost (or welfare) sharing problem where the constraints and objectives differ slightly from the classical approach taken in cooperative game theory. Nonetheless, many established cost sharing methodologies provide performance guarantees for the class of resource allocation problems considered in this paper. The first such protocol is known as the *marginal contribution* protocol [2], [40] and takes on the form

$$f_r^{\rm MC}(i, S) := W_r(S) - W_r(S \setminus \{i\}). \tag{8}$$

for any agent i and subset of agents $S \subseteq N$. The second such protocol is known as the *weighted Shapley value* [34], [50], [51]. The weighted Shapley value is a generalization of the Shapley Value where each agent is associated with a positive weight $\omega_i \in \mathbb{R}_+$. The weighted Shapley value protocol takes on the following form: for any resource $r \in \mathcal{R}$ and agent set $S \subseteq N$

$$f_r^{\rm WSV}(i, S; \omega) := \sum_{T \subseteq S : i \in T} \frac{\omega_i}{\sum_{j \in T} \omega_j} \left(\sum_{R \subseteq T} (-1)^{|T| - |R|} W_r(R) \right). \tag{9}$$

The Shapley value is captured if $w_i = 1$ for all agents $i \in N$. The properties associated with each protocol are highlighted in Table I.

Example 2.1: Let $N = \{1, 2, 3\}$. Consider any welfare function $W_r : 2^N \to R$. The distributed shares according to the marginal contribution protocol are

$$\begin{aligned} f_r^{\mathrm{MC}}(1, N) &= W_r(\{1, 2, 3\}) - W_r(\{2, 3\}) \\ f_r^{\mathrm{MC}}(2, N) &= W_r(\{1, 2, 3\}) - W_r(\{1, 3\}) \\ f_r^{\mathrm{MC}}(3, N) &= W_r(\{1, 2, 3\}) - W_r(\{1, 2\}). \end{aligned}$$

The distributed shares for player 1 according to the Shapley value protocol ($\omega_i = 1$ for each agent) are

$$\begin{aligned} f_r^{\mathrm{SV}}(1, N) &= \frac{1}{3}\left(W_r(\{1, 2, 3\}) - W_r(\{2, 3\})\right) + \frac{1}{6}\left(W_r(\{1, 2\}) - W_r(\{2\})\right) \\ &\quad + \frac{1}{6}\left(W_r(\{1, 3\}) - W_r(\{3\})\right) + \frac{1}{3}\left(W_r(\{1\}) - W_r(\{\emptyset\})\right) \end{aligned}$$

The distributed shares for the remaining agents are determined in a similar fashion. Note that the marginal contribution protocol is not budget-balanced while the (weighted) Shapley value protocol is budget-balanced.

III. The Limitations of Game Theoretic Designs

In this section we focus on the viability of establishing systematic methodologies for the design of budget-balanced protocols that meet our desired objectives. Unfortunately, the forthcoming results prove that meeting these objectives is computationally prohibitive.

A. A Characterization of Scalable Protocols

Our first result provides a complete characterization of all protocols that are scalable, budget-balanced, and guarantee the existence of an equilibrium for any resource allocation problem.

Definition 3.1 (Budget-balanced): A protocol $\{f_r\}$ is budget-balanced if for any resource $r \in \mathcal{R}$ and any agent set $S \subseteq N$, $\sum_{i \in S} f_r(i, S) = W_r(S)$.

Theorem 3.1: Let $\mathcal{G}$ be the set of distributed welfare games. A protocol is scalable, budget-balanced, and guarantees the existence of any equilibrium for any game $G \in \mathcal{G}$ if and only if the protocol is a weighted Shapley value.

An alternative interpretation of Theorem 3.1 is the following: If a protocol is scalable, budget-balanced, and does not represent a weighted Shapley value, then there exists a game $G \in \mathcal{G}$ such that an equilibrium does not exist. Accordingly, the only "universal" methodology that guarantees the existence of an equilibrium in any game through the use of budget-balanced protocols is the weighted Shapley value. The value of this characterization is the identification of the *complete design space* that a system designer needs to consider which entails just variations in player weights $\{\omega_i\}_{i \in N}$. The limitation of this result hinges on the fact that in many settings computing a weighted Shapley value is intractable [52].[11] It remains an open question to understand how weaker conditions on budget-balanced protocols impact this characterization.

We present the proof of Theorem 3.1 in the Appendix. The proof follows a similar structure to the proof presented in [39] which proves a parallel result for a class of cost minimization problems. It is important to point out that welfare maximization problems and cost minimization problems are not one in the same. There are several results for cost minimization problems [28], [54] that do not extend to welfare maximization problems and vice versa.

B. Efficiency Limitations in Budget-Balanced Protocols

In this section we provide a second limitation of utility design for engineering systems by proving that it is impossible to guarantee a price of stability of 1 across all games when restricting attention to budget-balanced protocols. This is in contrast to the marginal contribution protocol in (8) which always guarantees that the allocation that maximizes the global welfare W is an equilibrium but it not budget-balanced.

Theorem 3.2: Let $\mathcal{G}$ be the set of distributed welfare games with submodular welfare functions and a fixed weighted Shapley value protocol. The price of stability across the set of games $\mathcal{G}$ is equal to $1/2$.

The proof of Theorem 3.2 is included in the Appendix.

IV. Addressing the limitations of game theoretic designs

The previous section established two theoretical limitations of utility design for distributed engineering systems when restricting attention to the framework of finite strategic form games. In this section, we seek to overcome these limitations by conditioning protocols on additional information, i.e., a state. To do this, we focus on a class of protocols termed *ordered protocols*[12],

[11] It is important to highlight that there are approximation methods for computing Shapley values [53]; however, such approximations are not guaranteed to ensure the existence of an equilibrium for all games $G \in \mathcal{G}$.

[12] In the economic literature an ordered protocol is commonly referred to as *incremental cost sharing*. We choose to use the terminology ordered protocol to remain consistent with [39].

which represent a limiting case of the weighted Shapley value. We then introduce a dynamic adjustment process for these ordered protocols that can be utilized to overcome the aforementioned limitations.

A. Properties of Ordered Protocols

Before formally defining an ordered protocol, we introduce the following notation. Let $X(N)$ represent the set of possible orders (or permutations) of the agent set N. Therefore, an order $x \in X(N)$ is a vector of length n where each entry of x is associated with a unique entry in $\{1, ..., n\}$. Let x_i represent the index of agent i in the order x. For any agent set $S \subseteq N$ and agent $i \in S$ define $N_i(x, S) := \{j \in S : x_j \leq x_i\}$ as the set of agents with an index less than the index of agent i.

Definition 4.1 (Ordered Protocol): Define an order $x_r \in X(N)$ for each resource $r \in \mathcal{R}$. An ordered protocol takes on the following form: for any resource $r \in \mathcal{R}$, agent set $S \subseteq N$, and agent $i \in S$

$$f_r^{\mathrm{ORD}}(i, S; x_r) := W_r\left(N_i\left(x_r, S\right)\right) - W_r\left(N_i\left(x_r, S \setminus \{i\}\right)\right). \tag{10}$$

An ordered protocol assigns each agent a distributed share in accordance with their marginal contribution to the welfare in their respective order as illustrated in the following example.

Example 4.1 (Ordered Protocols): Let $N = \{1, 2, 3\}$. Consider any welfare function $W_r : 2^N \to R$. Suppose we have the ordering x where $x_1 = 1$, $x_2 = 3$, and $x_3 = 2$, then the distributed shares for the full set N are

$$\begin{aligned} f_r^{\mathrm{ORD}}(1, N; x) &= W_r(\{1\}) - W_r(\emptyset) \\ f_r^{\mathrm{ORD}}(3, N; x) &= W_r(\{1, 3\}) - W_r(\{1\}) \\ f_r^{\mathrm{ORD}}(2, N; x) &= W_r(\{1, 2, 3\}) - W_r(\{1, 3\}) \end{aligned}$$

The distributed shares for the subset $S = \{2, 3\}$ are

$$\begin{aligned} f_r^{\mathrm{ORD}}(3, S; x) &= W_r(\{3\}) - W_r(\emptyset) \\ f_r^{\mathrm{ORD}}(2, S; x) &= W_r(\{2, 3\}) - W_r(\{3\}) \end{aligned}$$

The distributed shares for the remaining agent sets are determined in a similar fashion.

Consider the special case where $W_r(S) = 1$ for all $S \subseteq \{1, 2, 3\}$ and $W_r(\emptyset) = 0$. The distributed shares for all considered protocols given the full player set N are:

Marginal contribution	*Shapley value*	*Ordered*
$f_r^{\mathrm{MC}}(1, N; x) = 0$	$f_r^{\mathrm{SV}}(1, N; x) = 1/3$	$f_r^{\mathrm{ORD}}(1, N; x) = 1$
$f_r^{\mathrm{MC}}(2, N; x) = 0$	$f_r^{\mathrm{SV}}(2, N; x) = 1/3$	$f_r^{\mathrm{ORD}}(2, N; x) = 0$
$f_r^{\mathrm{MC}}(3, N; x) = 0$	$f_r^{\mathrm{SV}}(3, N; x) = 1/3$	$f_r^{\mathrm{ORD}}(3, N; x) = 0$

It is straightforward to verify that any ordered protocol with order x is budget-balanced, i.e., for any agent set $S \subseteq N$

$$\sum_{i \in S} f_r^{\mathrm{ORD}}(i, S; x) = W_r(S)$$

as the intermediate terms cancel out in the above summation. Furthermore, note that an agent's distributed share is unaffected by the presence of agents with higher indices. Lastly, an ordered protocol with order x is equivalent to a weighted Shapley value. To see this, define for each agent i a sequence of positive weights $\{\omega_i^k\}_{k=1}^{\infty}$ with the following property: for any agents $i, j \in N$

$$x_i < x_j \;\Leftrightarrow\; \lim_{k \to \infty} \left(\frac{\omega_i^k}{\omega_j^k} \right) \to 0$$

One particular choice of weights that satisfy this condition is setting $\omega_i^k = (x_i)^k$. It is straightforward to verify the following equivalence

$$f_r^{\mathrm{ORD}}(i, S; x) = \lim_{k \to \infty} f_r^{\mathrm{WSV}}(i, S; \{\omega_i^k\}).$$

The fact that an ordered protocol is equivalent to a weighted Shapley value gives us the following corollary from Theorems 3.1 and 3.2.

Corollary 4.1: Let $\mathcal{G}$ be the set of distributed welfare games with agent set N, submodular welfare functions, and an ordered protocol with a fixed order $x \in X(N)$, i.e., for any game $G \in \mathcal{G}$ and resource $r \in \mathcal{R}$ we have $x_r = x$. The ordered protocol guarantees the existence of an equilibrium for any game. Furthermore, the price of anarchy and price of stability across the set of games $\mathcal{G}$ is equal to $1/2$.

Interestingly, the following lemma states that if we are able to condition an ordered protocol on the game structure, i.e., no longer requiring a fixed ordering across all resources (scalability), then we can ensure that the optimal allocation is in fact an equilibrium.

Lemma 4.1: Let G be any distributed welfare games with submodular welfare functions. There exists a set of ordered protocols with orders $\{x_r\}$ such that the optimal allocation is an equilibrium.

Proof: We prove this lemma by constructing an order for each resource $\{x_r\}$ such that the ordered protocol ensures that the optimal allocation is an equilibrium. Let $a^{\rm opt}$ be an optimal allocation. Define the following order: for each resource $r \in \mathcal{R}$, assign each agent $i \in \{a^{\rm opt}\}_r$ a unique entry in $\{1, ..., |a^{\rm opt}|_r\}$ and assign each agent $j \notin \{a^{\rm opt}\}_r$ a unique entry in $\{|a^{\rm opt}|_r + 1, ..., n\}$.

Suppose $a^{\rm opt}$ is not an equilibrium given the ordered protocol conditioned on $\{x_r\}$. Then this means that there exists an agent $i \in N$ with an action $a_i \in \mathcal{A}_i$ such that

$$U_i(a_i, a_{-i}^{\rm opt}) > U_i(a^{\rm opt}).$$

Without loss of generalities assume $a_i \cap a_i^{\rm opt} = \emptyset$. Because the welfare functions are submodular for each resource $r \in a_i^{\rm opt}$ we have that

$$\begin{aligned} f_r^{\rm ORD}(i, \{a^{\rm opt}\}_r; x_r) &= W_r\left(N_i\left(x_r, \{a^{\rm opt}\}_r\right)\right) - W_r\left(N_i\left(x_r, \{a^{\rm opt}\}_r \setminus \{i\}\right)\right) \\ &\geq W_r(\{a^{\rm opt}\}_r) - W_r(\{a^{\rm opt}\}_r \setminus \{i\}), \end{aligned}$$

meaning that any agent's utility is greater than or equal to the agent's true marginal contribution. This follows from the definition of submodularity since $N_i\left(x_r, \{a^{\rm opt}\}_r\right) \subseteq \{a^{\rm opt}\}_r$. This implies

$$\begin{aligned} W(\emptyset, a_{-i}^{\rm opt}) &\geq W(a^{\rm opt}) - \sum_{r \in a_i^{\rm opt}} f_r^{\rm ORD}(i, \{a^{\rm opt}\}; x_r) \\ &= W(a^{\rm opt}) - U_i(a^{\rm opt}). \end{aligned} \tag{11}$$

Because of the defined orderings, we have that for any $r \in a_i$

$$f_r^{\rm ORD}(i, \{a_i, a_{-i}^{\rm opt}\}_r; x_r) = W_r(\{a_i, a_{-i}^{\rm opt}\}_r) - W_r(\{a_i, a_{-i}^{\rm opt}\}_r \setminus \{i\}).$$

which implies that

$$\begin{aligned} W(a_i, a_{-i}^{\rm opt}) &= W(\emptyset, a_{-i}^{\rm opt}) + \sum_{r \in a_i} f_r^{\rm ORD}(i, \{a_i, a_{-i}^{\rm opt}\}_r; x_r) \\ &= W(\emptyset, a_{-i}^{\rm opt}) + U_i(a_i, a_{-i}^{\rm opt}). \end{aligned} \tag{12}$$

Combining (11) and (12) gives us

$$U_i(a_i, a_{-i}^{\rm opt}) > U_i(a^{\rm opt}) \;\Rightarrow\; W(a_i, a_{-i}^{\rm opt}) > W(a^{\rm opt})$$

which is a contradiction by the optimality of $W(a^{\rm opt})$. ■

B. Dynamic Adjustment of Ordered Protocols

The previous section demonstrates that both of these limitations can be overcome using ordered protocols if the system designer (i) has knowledge of the specific game structure and (ii) is capable of evaluating the optimal allocation a priori. Since both of these conditions are infeasible and lack scalability, we introduce a dynamic adjustment process for ordered protocols that eliminates these needs while providing the same guarantees irrespective of the game structure. This dynamic adjustment process can be thought of in two lights. One interpretation of this process is as an intelligent approach to utility design. The second interpretation could be as a side algorithm used to improve the limiting behavior, i.e., equilibrium selection.

Defining this dynamic process requires a notion of time in our problem setup. To that end, let $a(0)$, $a(1)$, $a(2)$, ..., represent a sequence of allocations generated by a given decision making process. The action profile at time t is a tuple $a(t) := (a_1(t), ..., a_n(t))$ consisting of the action of each agent at time t. Similarly, we denote the order for resource r at time t as $x_r(t)$ and the collection of orders as $x(t) = \{x_r(t)\}$. We adopt the convention that the ordering at time t, $x(t)$, represents an ordering for the allocation at time $t-1$, i.e., for each resource $r \in \mathcal{R}$ we have $x_r(t) \in X(\{a(t-1)\}_r)$. For convenience, we define this order $x_r(t)$ over the set $X(\{a(t-1)\}_r)$ as opposed to being an order over $X(N)$ as described in the previous section. The definition of an ordered protocol in (10) immediately extends to this case in the logical manner.

The core of our design is the establishment of a dynamic process by which the orders are updated in a local fashion. To that end, define an initial order for each resource $r \in \mathcal{R}$ as $x_r(1) \in X(\{a(0)\}_r)$. The ordering for any resource $r \in \mathcal{R}$ evolves according to a stochastic transition function of the following form: for each $t \geq 1$

$$x_r(t+1) = P_r(x_r(t), \{a(t)\}_r).$$

To formally state the dynamics we introduce a bit of notation. Let $x_{r,i}(t) \in \{1, 2, ..., |a(t-1)|_r\}$ represent the order of the i^{th} agent in the order $x_r(t)$. The order for each resource $r \in \mathcal{R}$ evolves according to the following two rules:

(i) Associate with each agent $j \in Z = \{a(t-1)\}_r \cap \{a(t)\}_r$ a unique order $x_{r,j}(t) \in \{1, ..., |Z|\}$ according to the following condition: For any two agent $i, j \in Z$

$$x_{r,i}(t-1) < x_{r,j}(t-1) \;\Rightarrow\; x_{r,i}(t) < x_{r,j}(t).$$

Notice that this condition enforces a unique order over the agent set Z.

(ii) Associate with each agent $j \in \{a(t)\}_r \setminus \{a(t-1)\}_r$ a unique order $x_{r,j}(t) \in \{|Z| + 1, ..., |\{a(t)\}_r|\}$ according to any deterministic or stochastic rule.

We express the ordering dynamics for all resource by the state transition function

$$x(t+1) = P(x(t), a(t)).$$

Note that these dynamics satisfy the property

$$a(t) = a(t-1) \;\Rightarrow\; x(t+1) = x(t). \tag{13}$$

We now introduce a notion of state based utility functions. Let x be an order for the allocation a, i.e., $x_r \in X(\{a\}_r)$ for each resource $r \in \mathcal{R}$. We refer to the set of orderings x as the state of the system. The utility for agent i given this action state pair is

$$U_i(a; x) = \sum_{r \in a_i} f_r(i, \{a\}_r; x_r). \tag{14}$$

We extend (14) to all allocations a and ordering x as follows:

$$\begin{aligned} U_i(a; x) &= U_i(a; x') \\ &= \sum_{r \in a_i} f_r(i, \{a\}_r; x'_r) \end{aligned}$$

where $x' = P(x, a)$ is the new ordering chosen according to our specified dynamic process and hence is a valid ordering for the allocation a. Accordingly, we focus on the following set of equilibria defined as follows:

Definition 4.2 (Equilibria of the Dynamic Process): The equilibria of the dynamic order adjustment process is the set of all action state pairs $[a^*, x^*]$ such that the following conditions hold:

(i) For every resource $r \in \mathcal{R}$, we have $x^*_r \in X(\{a^*\}_r)$.

(ii) For every agent $i \in N$, we have $U_i(a^*_i, a^*_{-i}; x^*) = \max_{a_i \in \mathcal{A}_i} U_i(a_i, a^*_{-i}; x^*)$.

Much like pure Nash equilibria, the defined equilibria represent the set of fixed points of a myopic Cournot adjustment process where at each instance of time an agent chooses the following action

$$a_i(t) \in \arg\max_{a_i \in \mathcal{A}_i} U_i(a_i, a_{-i}(t-1); x(t)).$$

Here the term myopic refers to the fact that the future state trajectory and actions of agents do not impact the agents' decision making process. In social systems, this form of dynamics would require further justification. However, in engineering systems where the decision makers are programmable components, such equilibria are a valid solution concept.

Much like the previous analysis, we focus on whether our dynamic process ensures the existence and efficiency of such equilibria. For efficiency, we extend the definition of price of anarchy and price of stability over in the logical manner.

Theorem 4.1: Let $\mathcal{G}$ as the set of distributed welfare games with submodular welfare functions and dynamic ordered protocols. The set of equilibria of the dynamic process is nonempty for any game $G \in \mathcal{G}$. Furthermore, the price of anarchy is $1/2$ and the price of stability is 1 across the set of games $\mathcal{G}$.

Proof: We prove this results by demonstrating that our dynamic ordering adjustment process gives rise to a state based potential game [45]. To show this let $a \in \mathcal{A}$ represent any allocation and let $x \in X$ represent an order for the allocation a. For any agent $i \in N$ and action profile $a_i' \in \mathcal{A}$ we have

$$U_i(a_i', a_{-i}; x) - U_i(a_i, a_{-i}; x) = \sum_{r \in a_i' \setminus a_i} f_r(i, \{a\}_r \cup \{i\}; x_r') - \sum_{r \in a_i \setminus a_i'} f_r(i, \{a\}_r; x_r) \tag{15}$$

where the order $x' = P(x, a)$. Focusing on the first term we have

$$\begin{aligned} \sum_{r \in a_i' \setminus a_i} f_r(i, \{a\}_r \cup \{i\}; x_r') &= \sum_{r \in a_i' \setminus a_i} (W_r(\{a\}_r \cup \{i\}) - W_r(\{a\}_r)) && (16) \\ &= W(a_i', a_{-i}) - W(a_i'', a_{-i}) && (17) \end{aligned}$$

where $a_i'' = a_i' \cap a_i$. This is true because agent i is last in the order x_r' for each resource $r \in a_i' \setminus a_i$. Focusing on the second set of terms we have

$$\begin{aligned} \sum_{r \in a_i \setminus a_i'} f_r(i, \{a\}_r; x_r) &\geq \sum_{r \in a_i \setminus a_i'} (W_r(\{a\}_r) - W_r(\{a\}_r \setminus \{i\})) && (18) \\ &= W(a_i, a_{-i}) - W(a_i'', a_{-i}) && (19) \end{aligned}$$

because of the submodularity of W. Therefore, combining the two set of equations gives us

$$U_i(a_i', a_{-i}; x) - U_i(a_i, a_{-i}; x) \leq W(a_i', a_{-i}) - W(a). \tag{20}$$

Consider any optimal allocation a^* with an ordering x^*. Suppose $[a^*, x^*]$ is not an equilibrium. This means that there exists an agent $i \in N$ with an action $a_i \in \mathcal{A}_i$ such that

$$U_i(a_i, a_{-i}^*; x^*) > U_i(a^*; x^*).$$

By (20), this also implies that

$$W(a_i, a_{-i}^*) > W(a^*)$$

which is a contradiction by the optimality of a^*. Therefore, the equilibrium set is not empty and furthermore the price of stability is 1.

For the price of anarchy result, let $[a^{\rm ne}, x^{\rm ne}]$ represent any equilibrium. Accordingly, we know that for any optimal allocation $a^{\rm opt}$ we have

$$\begin{aligned} U_i(a_i^{\rm ne}, a_{-i}^{\rm ne}; x^{\rm ne}) &\geq U_i(a_i^{\rm opt}, a_{-i}^{\rm ne}; x^{\rm ne}) && (21)\\ &\geq W(a_i^{\rm opt}, a_{-i}^{\rm ne}) - W(\emptyset, a_{-i}^{\rm ne}) && (22)\end{aligned}$$

With a slight abuse of notation, let $W(a^{\rm ne}, a^{\rm opt})$ represent the welfare associated with $2n$ where n agents choose $a^{\rm ne}$ and n other agents choose $a^{\rm opt}$. Since W is increasing, we have that $W(a^{\rm opt}) \leq W(a^{\rm ne}, a^{\rm opt})$. We can evaluate $W(a^{\rm ne}, a^{\rm opt})$ as follows:

$$\begin{aligned} W(a^{\rm ne}, a^{\rm opt}) &= W(a^{\rm ne}) + \sum_{k=1}^{n} \big(W(a^{\rm ne}, a_{1:k}^{\rm opt}) - W(a^{\rm ne}, a_{1:k-1}^{\rm opt}) \big) \\ &\leq W(a^{\rm ne}) + \sum_{k=1}^{n} \big(W(a_k^{\rm opt}, a_{-k}^{\rm ne}) - W(\emptyset, a_{-k}^{\rm ne}) \big)\end{aligned}$$

where $a_{1:k}^{\rm opt} := (a_1^{\rm opt}, ..., a_k^{\rm opt})$ and the second inequality follows from the submodularity of W. Accordingly, we can derive the following bound on $W(a^{\rm opt})$ as

$$\begin{aligned} W(a^{\rm opt}) &\leq W(a^{\rm ne}, a^{\rm opt}) && (23)\\ &\leq W(a^{\rm ne}) + \sum_{i \in N} \big(W(a_i^{\rm opt}, a_{-i}^{\rm ne}) - W(\emptyset, a_{-i}^{\rm ne}) \big) && (24)\\ &\leq W(a^{\rm ne}) + \sum_{i \in N} U_i(a_i^{\rm opt}, a_{-i}^{\rm ne}; x^{\rm ne}) && (25)\\ &\leq W(a^{\rm ne}) + \sum_{i \in N} U_i(a_i^{\rm ne}, a_{-i}^{\rm ne}; x^{\rm ne}) && (26)\\ &= 2W(a^{\rm ne}), && (27)\end{aligned}$$

where the last equality comes from the fact that our protocol is budget-balanced. This completes the proof. ■

V. Concluding remarks

This paper focuses on testing the viability of the framework of potential games as an underlying framework for the design and control of multiagent systems. We derive two fundamental limitations of potential games which suggests that the framework of potential games is not rich enough to undertake the challenges inherent to multiagent system design. A key question for the progression of this game theoretic approach to multiagent systems entails identifying a game structure, similar to that of potential games, which is broad enough to handle these inherent challenges. Once such a mediating layer is identified, tools can be developed for both game design and learning design which could be used to tackle a wide variety of challenges in existing and future multiagent systems.

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VI. Appendix

A. Proof of Theorem 3.1

We start with an outline of the proof. The "if" direction of the proof establishes that the weighted Shapley value protocol guarantees that the resulting game is a weighted potential game irrespective of the specific resource allocation problem. Since an equilibrium is guaranteed to exist in any weighted potential game, this completes the proof this direction.

The "only if" direction of the proof is more involved. In particular, we identify a particular class of resource allocation problems such that the only budget-balanced protocol that guarantees the existence of an equilibrium is a weighted Shapley value. Hence, weighted Shapley values represent the only universal methodologies that guarantee the existence of an equilibrium across any game. The particular class of distributed welfare games that we consider consists of resource specific welfare functions of the form

$$W_r(S) = \begin{cases} 0 & S = \emptyset \\ v_r & S \neq \emptyset \end{cases} \tag{28}$$

where $v_r \geq 0$ is referred to as the value of resource r. Since this class of distributed welfare games only possess resource specific welfare functions of the form (28), our definition of scalability directly implies that the underlying protocols take of the following form: for any agent set $S \subseteq N$ and agent $i \in S$

$$f_r(i, S) = v_r \cdot f(i, S) \tag{29}$$

where $f : N \times 2^N \to R^+$ is the base protocol. To simplify notation, we write a resource specific distribution only in terms of $f(\cdot)$ and drop the resource specific indices.[13]

To prove this result we establish several sufficient conditions necessary for guaranteeing the existence of an equilibrium through a series of counterexamples. These counterexamples amount to choosing the number of agents N, the resource set $\mathcal{R}$, the respective values $\{v_r\}$, and the associated action sets $\{\mathcal{A}_i\}$. Effectively, these sufficient conditions eliminate any protocol that is not a weighted Shapley value and hence give us our desired result. The three main milestones of the proof are as follows:

1) If a protocol is scalable, budget-balanced, and guarantees the existence of an equilibrium in any game $G \in \mathcal{G}$ then the protocol must be *monotonic* which implies that $f(i, S) \geq f(i, T)$ whenever $i \in S \subseteq T \subseteq N$.[14]

[13]This simplification can easily be seen by grouping resources together to get the desired v_r, i.e., if for any agent $i \in N$ and action $a_i \in \mathcal{A}_i$, if $r \in a_i$ then this implies that $r' \in \mathcal{A}_i$. This procedure is equivalent to creating a new resource r'' with value $v_{r''} = v_r + v_{r'}$. Hence, if $v_r = v_{r'}$, then the protocol associated with this new resource with value $2v_r$ must satisfy the stated form. For simplicity, assume throughout that we have a family of resources each with value 1. By scalability, each of the resources must possess the same the same protocol $f(\cdot)$. If we group x resources together as highlighted above, then this new combined resource is identical to a single resource with value x and protocol $xf(\cdot)$.

[14]This current definition of monotonicity relies on the definition of the welfare function which imply that $W_r(S) = W_r(T)$ for any nonempty agent sets $S, T \subseteq N$.

2) If a protocol is scalable, budget-balanced, and guarantees the existence of an equilibrium in any game $G \in \mathcal{G}$ then the protocol is uniquely determined by $n-1$ pairwise distributed shares $f(1,\{1,2\})$, $f(1,\{1,3\})$, ..., $f(1,\{1,n\})$.
3) If a protocol is scalable, budget-balanced, and guarantees the existence of an equilibrium in any game $G \in \mathcal{G}$ then the protocol is a weighted Shapley value.

The outline of the proof is very similar to the proof set forth in [39], which also focuses on the specific welfare (or cost) functions of the form in (28). A key difference between the two approaches is the fact that [39] focuses on the problem of cost minimization in network formation versus our focus on welfare maximization. Note that utilizing objective functions of the form (28) yields very different phenomena in these different domains as welfare maximization encourages agents to spread out while cost minimization encourages agents to group together. While our proof follows a similar outline to the one set forth in [39], many of the of the steps require different arguments and counterexamples because of the variation in the problem formulation.

Proof of the "if" direction: First we prove the "if" direction. That is, the weighted Shapley value results in a protocol that is scalable, budget-balanced, and guarantees the existence of any equilibrium for any game $G \in \mathcal{G}$. Without loss of generality, we assume that we have no dummy agents. That is, for any agent $i \in N$ and any resource $r \in \mathcal{R}$, there exists an agent set $S \subseteq N \setminus \{i\}$ such that $W_r(S \cup \{i\}) > W_r(S)$. For the weighted Shapley value, the protocol takes on the following form: for any resource $r \in \mathcal{R}$, agent set $S \subseteq N$, and positive weight vector $\omega = \{\omega_i\}_{i \in N}$,.

$$f_r^{\mathrm{WSV}}(i,S;\omega) := \sum_{T \subseteq S : i \in T} \frac{\omega_i}{\sum_{j \in T} \omega_j} \left(\sum_{R \subseteq T} (-1)^{|T|-|R|} W_r(R) \right). \tag{30}$$

It is straightforward to see that the weighted Shapley value protocol is scalable. It is also budget-balanced as shown in [2].

To show that an equilibrium always exists we show that the weighted Shapley value protocol results in a weighted potential game. To see this, define for each resource $r \in \mathcal{R}$ a potential function $\phi_r^{\mathrm{WSV}} : 2^N \rightarrow R^+$ according to the following set based recursions

$$\begin{aligned} \phi_r^{\mathrm{WSV}}(\emptyset) &= 0 \\ \phi_r^{\mathrm{WSV}}(S) &= \frac{1}{\sum_{i \in S} w_i} \left[W_r(S) + \sum_{i \in S} w_i \phi_r^{\mathrm{WSV}}(S \setminus \{i\}) \right]. \end{aligned}$$

It is shown in [34] that this potential function directly translates to our protocol as follows

$$f_r^{\mathrm{WSV}}(i,S;\omega) = \omega_i \left(\phi_r^{\mathrm{WSV}}(S) - \phi_r^{\mathrm{WSV}}(S \setminus \{i\})\right).$$

Let $\phi(a) := \sum_{r\in\mathcal{R}} \phi_r(\{a\}_r)$. It is straightforward to show that

$$U_i(a_i,a_{-i}) - U_i(a_i',a_{-i}) = \omega_i\left(\phi(a_i,a_{-i}) - \phi(a_i',a_{-i})\right)$$

meaning that the game is a weighted potential game with potential function ϕ and agent weights $w_i = 1/\omega_i$. Therefore, an equilibrium is guaranteed to exist in any game $G \in \mathcal{G}$ since any allocation maximizing the potential function ϕ is an equilibrium.

Proof of the "only if" direction: The rest of this section is dedicated to showing the "only if" direction. To simplify the forthcoming proof we focus initially on positive protocols and then illustrate how positivity can be relaxed using arguments that parallel [39]. A protocol $\{f_r\}$ is positive if for any agent set $S \subseteq N$ and agent $i \in S$

$$W_r(S) > 0 \ \Rightarrow\ f_r(i,S) > 0.$$

The proof of this direction encompasses proving the three milestones highlighted above.

Step #1: Monotonicity of Protocols: The following lemma establishes a set of inequalities that a protocol must satisfy to guarantee the existence of an equilibrium. Part of the inequalities focus on protocols that are monotonic. At this stage, we have not proved the importance of monotonicity; however, that will be proved in the ensuing lemma.

Lemma 6.1: Let f be a protocol that is scalable, budget-balanced, positive, and guarantees the existence of an equilibrium in any game $G \in \mathcal{G}$. Let $S \subset N$ be a nonempty agent set and i and j be distinct agents not in S. Then

$$f(i,S\cup\{i\}) < f(i,S\cup\{i,j\}) \ \Rightarrow\ f(j,S\cup\{j\}) \leq f(j,S\cup\{i,j\}). \tag{31}$$

Furthermore, if the protocol f is monotonic then we have the stronger result of

$$\begin{aligned}(f(i,S\cup\{i\}) - f(i,S\cup\{i,j\}))\cdot(f(j,\{j\}) - f(j,\{i,j\})) = \\ (f(j,S\cup\{j\}) - f(j,S\cup\{i,j\}))\cdot(f(i,\{i\}) - f(i,\{i,j\}))\end{aligned} \tag{32}$$

Proof: Before proving this lemma, for notational simplicity we denote the four differences in (32) by Δ_1, Δ_2, Δ_3, and Δ_4 respectively. In terms of this notation, the first part of this lemma establishes that

$$\Delta_1 < 0 \ \Rightarrow\ \Delta_3 \leq 0.$$

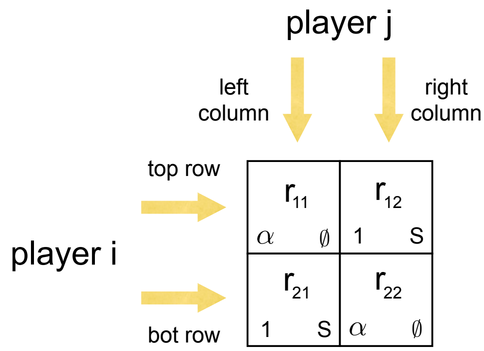

Fig. 1. Illustration of Game to Prove Lemma 6.1

To prove this we consider the following game as highlighted in Figure 1. This game entails the resource set $\mathcal{R} = \{r_{11}, r_{12}, r_{21}, r_{22}\}$, relative values $v_{11} = v_{22} = \alpha > 0$ and $v_{12} = v_{21} = 1$, and action sets

$$\begin{aligned} \mathcal{A}_i &= \{T = \{r_{11}, r_{12}\}, B = \{r_{21}, r_{22}\}\} \\ \mathcal{A}_j &= \{L = \{r_{11}, r_{21}\}, R = \{r_{12}, r_{22}\}\}. \end{aligned}$$

In this setting agents 1 and 2 have asymmetric action sets. There is also a set of agents S with a fixed action, i.e., the action set for any agent $k \in S$ is $\mathcal{A}_k = \{Z = \{r_{12}, r_{21}\}\}$ meaning that each agent $j \in S$ only has one action so the agent is fixed. The joint action set can be expressed as $\mathcal{A} = \{TL, TR, BL, BR\}$ as we omit mentioning the agents in S.

If the protocol guarantees the existence of an equilibrium for any game $G \in \mathcal{G}$, then for any choice of S and α one of the 4 joint actions must be an equilibrium. If the allocation (T, L) is an equilibrium then both of the following inequalities must be satisfied

$$\begin{aligned} U_i(T, L) &\geq U_i(B, L) \\ U_j(T, L) &\geq U_j(T, R) \end{aligned}$$

where the first entry represents the action for agent i and the second entry represents the action for agent j. In terms of our protocol f this translates to *both* of the following inequalities being satisfied

$$\alpha f(i, \{i, j\}) + f(i, S \cup \{i\}) \geq f(i, S \cup \{i, j\}) + \alpha f(i, \{i\}) \quad (33)$$

$$\alpha f(j, \{i, j\}) + f(j, S \cup \{j\}) \geq f(j, S \cup \{i, j\}) + \alpha f(j, \{j\}) \quad (34)$$

Rewriting (33) and (34) in terms of Δ_1, Δ_2, Δ_3, and Δ_4, we have

$$\alpha\Delta_4 \leq \Delta_1 \tag{35}$$

$$\alpha\Delta_2 \leq \Delta_3 \tag{36}$$

By symmetry, it is straightforward to verify that the conditions are also necessary to ensure (B, R) is an equilibrium. If (T, R) (or equivalently (B, L)) is an equilibrium then *both* of the following inequalities must be satisfied

$$\alpha\Delta_4 \geq \Delta_1 \tag{37}$$

$$\alpha\Delta_2 \geq \Delta_3 \tag{38}$$

Since $f(\cdot)$ is positive and budget-balanced, we have that $\Delta_2, \Delta_4 \geq 0$ resulting from the fact that $f(i, \{i\}) = f(j, \{j\}) = 1$. If $\Delta_1 < 0$, then for all $\alpha \geq 0$ the joint action profile (T, R) must be an equilibrium as (35) can never be satisfied. Therefore, (38) must be satisfied for all $\alpha \geq 0$ which implies that $\Delta_3 \leq 0$. This establishes the first result in (31).

For the second results we focus on the special case when the protocol f is monotonic which implies that $\Delta_1, \Delta_3 \geq 0$. We prove that $\Delta_1\Delta_2 = \Delta_3\Delta_4$ holds by a case analysis.

- **Case 1:** Suppose $\Delta_2 = \Delta_4 = 0$. Then (32) holds trivially.
- **Case 2:** Suppose $\Delta_2 > 0$ while $\Delta_4 = 0$. For any $\Delta_3 \geq 0$ and sufficiently small α we have that (T, L) must be an equilibrium as we can ensure that (38) is not satisfied. Since (T, L) is an equilibrium this implies that $\Delta_1 = 0$ so (32) holds.
- **Case 3:** Suppose $\Delta_2 = 0$ while $\Delta_4 > 0$. Symmetric argument to Case 2.
- **Case 4:** Suppose $\Delta_2 > 0$ while $\Delta_4 > 0$. For any $\alpha \leq \Delta_1/\Delta_4$ we know that the action profile (T, L) must be an equilibrium; hence (36) must also be satisfied which ensures that $\Delta_1/\Delta_4 \leq \Delta_3/\Delta_2$. For any $\alpha \geq \Delta_1/\Delta_4$ we know that the action profile (T, R) must be an equilibrium; hence (38) must also be satisfied which ensures that $\Delta_1/\Delta_4 \geq \Delta_3/\Delta_2$. This implies that $\Delta_1/\Delta_4 = \Delta_3/\Delta_2$ so (32) holds.

■

The following lemma establishes the first milestone of our proof by demonstrating the importance of monotonicity in protocols.

Lemma 6.2: Let f be a protocol that is scalable, budget-balanced, positive, and guarantees the existence of an equilibrium in any game $G \in \mathcal{G}$. Then the protocol f must be monotonic.

Proof: Note that if f is positive and budget-balanced then we have that for any agents $i, j \in N$

$$f(i, \{i\}) = 1 \geq f(i, \{i, j\})$$

therefore a protocol starts off monotonic. We prove this theorem by contradiction. Assume that f is not monotonic which implies that there exists an agent set $S \subseteq N$ and a pair of agent $i, i' \in S$ such that

$$f(i, S \setminus \{i'\}) < f(i, S). \tag{39}$$

Let S be a minimal set that satisfies the above inequality. Note that S must contain at least one additional agent $j \neq i, i'$ since a protocol starts off monotonic.

By Lemma 6.1, we have that (39) directly implies that

$$f(i', S \setminus \{i\}) \leq f(i', S).$$

Since the protocol is budget-balanced we have

$$\sum_{j \in S \setminus \{i\}} f(j, S \setminus \{i\}) + \sum_{j \in S \setminus \{i'\}} f(j, S \setminus \{i'\}) = 2 = \sum_{j \in S} f(j, S) + \sum_{j \in S \setminus \{i, i'\}} f(j, S \setminus \{i, i'\}).$$

Rearranging the above summation we have

$$f(i, S) + f(i', S) + \sum_{j \in S \setminus \{i, i'\}} [f(j, S) + f(j, S \setminus \{i, i'\})] =$$

$$f(i, S \setminus \{i'\}) + f(i', S \setminus \{i\}) + \sum_{j \in S \setminus \{i, i'\}} [f(j, S \setminus \{i\}) + f(j, S \setminus \{i'\})]$$

Since $f(i, S) > f(i, S \setminus \{i'\})$ and $f(i', S) \geq f(i', S \setminus \{i\})$, there exists an agent $j \in S \setminus \{i, i'\}$ with a distribution share that satisfies

$$f(j, S) + f(j, S \setminus \{i, i'\}) < f(j, S \setminus \{i\}) + f(j, S \setminus \{i'\}). \tag{40}$$

We now construct a game demonstrating that if the above conditions are satisfied then an equilibrium need not exist. Consider the following example with resources $\mathcal{R} = \{r_1, r_2, r_3, r_4\}$ and values $v_1 = v_2 = \alpha$ and $v_3 = v_4 = 1$ where $\alpha > 1$ and satisfies

$$\alpha(f(j, S) + f(j, S \setminus \{i, i'\})) < f(j, S \setminus \{i\}) + f(j, S \setminus \{i'\}). \tag{41}$$

Let $\bar{S} = S \setminus \{i, i', j\}$. Consider the situation where agent i' has one action $\{r_2, r_4\}$ and any agent $j \in \bar{S}$ also has one action $\{r_1, r_2, r_3, r_4\}$ meaning that these are fixed and have no other

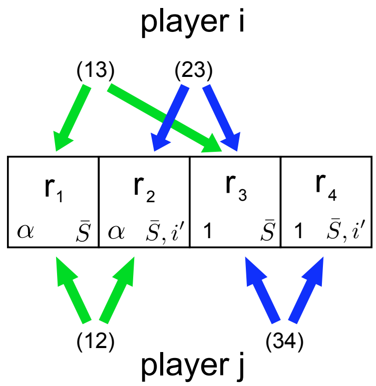

Fig. 2. Illustration of Game to Prove Lemma 6.2

options as depicted in Figure 2. This example boils down to a game between two agents i and j where each agent has two options: $\mathcal{A}_i = \{(13) = \{r_1, r_3\}, (23) = \{r_2, r_3\}\}$ and $\mathcal{A}_j = \{(12) = \{r_1, r_2\}, (34) = \{r_3, r_4\}\}$. There are four possible joint actions $\mathcal{A} = \{(13, 12), (13, 34), (23, 12), (23, 34)\}$ where the first entry in every action profile is agent i's action. We now demonstrate that an equilibrium does not exist by exploring the four possible joint actions.

- **Case 1:** $(23, 12) \rightarrow (23, 34)$: This transition is a better response for agent j if

$$f(j, S \setminus \{i'\}) + f(j, S \setminus \{i\}) \geq \alpha(f(j, S \setminus \{i, i'\}) + f(j, S))$$

  which is true because of (41). Hence the action profile $(23, 12)$ is not an equilibrium.
- **Case 2:** $(23, 34) \rightarrow (13, 34)$: This transition is a better response for agent i if

$$\alpha f(i, S \setminus \{i', j\}) \geq \alpha f(i, S \setminus \{j\})$$

  which is true because S was a minimal set by assumption. Hence the action profile $(23, 34)$ is not an equilibrium.
- **Case 3:** $(13, 34) \rightarrow (13, 12)$: This transition is a better response for agent j if

$$\alpha(f(j, S \setminus \{i'\}) + f(j, S \setminus \{i\})) \geq f(j, S \setminus \{i'\}) + f(j, S \setminus \{i\}))$$

  which is true because $\alpha > 1$. Hence the action profile $(13, 34)$ is not an equilibrium.
- **Case 4:** $(13, 12) \rightarrow (23, 12)$: This transition is a better response for agent i if

$$\alpha f(i, S) \geq \alpha f(i, S \setminus \{i'\})$$

  which is our assumption in (39). Hence the action profile $(13, 12)$ is not an equilibrium.

Therefore, an equilibrium does not exist. ■

Step #2: Pairwise Shares Uniquely Determine Protocol: The second milestone of the proof focuses on identifying the relationship between pairwise shares and the resultant protocol. These proofs build upon the results in Step #1, which establishes the necessity of monotonicity. The flavor of the proofs shift from counterexamples to more algebraic proofs. Accordingly, the proofs contained within this section are more in line with the proof in [39] for the case of cost minimization. In the case when the proofs are identical, we will direct the reader to the proof in [39] and give a short outline.

Lemma 6.3: Let f be a protocol that is scalable, budget-balanced, positive, and guarantees the existence of an equilibrium in any game $G \in \mathcal{G}$. For every three agents $i, j, k \in N$, the protocol satisfies

$$f(i, \{i, j\}) \times f(j, \{j, k\}) \times f(k, \{k, i\}) = f(j, \{i, j\}) \times f(k, \{j, k\}) \times f(i, \{k, i\}) \tag{42}$$

Proof: This proof of this lemma can be found in Lemma 5.8 of [39]. The proof is algebraic and utilized two main properties. The first property is budget-balanced which ensures that for any two agents $i, j \in N$

$$f(i, \{i, j\}) + f(j, \{i, j\}) = 1.$$

The second property is monotonicity which by Lemma 6.2 is necessary to guarantee the existence of an equilibrium for any game $G \in \mathcal{G}$. The property that is utilized in the proof stems Lemma 6.1 which ensures that for any monotonic protocol and any agents i, j, k we have

$$\begin{aligned}(f(i, \{i, k\}) - f(i, \{i, j, k\})) \cdot (f(j, \{j\}) - f(j, \{i, j\})) = \\ (f(j, \{j, k\}) - f(j, \{i, j, k\})) \cdot (f(i, \{i\}) - f(i, \{i, j\})).\end{aligned}$$

■

The second lemma in this section demonstrates that two protocols with identical pairwise distributed shares yield the same protocol. We define the set of pairwise distributed shares to agent i as $\{f(i, \{i, j\})\}_{j \in N \setminus \{i\}}$.

Lemma 6.4: Let f^1, f^2 be two protocols that are scalable, budget-balanced, positive, and guarantee the existence of an equilibrium in any game $G \in \mathcal{G}$. If f^1 and f^2 have identical pairwise distributed shares for some agent i, i.e., $f^1(i, \{i, j\}) = f^2(i, \{i, j\})$ for all agents $j \in N \setminus \{i\}$, then they have identical pairwise distributed shares for all agents.

Proof: Consider any two agents $j, k \in N \setminus \{i\}$. By positivity and budget-balanced, we can rewrite (42) as

$$\frac{f(j,\{j,k\})}{1-f(j,\{j,k\})} = \frac{1-f(k,\{j,k\})}{f(k,\{j,k\})} = \left(\frac{1-f(i,\{i,j\})}{f(i,\{i,j\})}\right)\left(\frac{f(i,\{k,i\})}{1-f(i,\{k,i\})}\right).$$

Therefore, the pairwise shares for both agents j and k are uniquely determined by the pairwise shares of agent i. ■

The next lemma establishes the second milestone of the proof by illustrating that any two protocols with identically distributed shares for any agent are identical protocols.

Lemma 6.5: Let f^1, f^2 be two protocols that are scalable, budget-balanced, positive, and guarantee the existence of an equilibrium in any game $G \in \mathcal{G}$. If f^1 and f^2 have identical pairwise distributed shares for some agent i, then they are identical protocols.

Proof: Let f be any protocol that is scalable, budget-balanced, positive, and guarantees the existence of an equilibrium in any game $G \in \mathcal{G}$. Let S represent an agent set with at least 3 agents. For every distinct agents $i, j \in S$, rewriting equation (32) (with $S \setminus \{i,j\}$ playing the role of S) gives us

$$f(j,S) = f(j, S\setminus\{i\}) + \frac{1-f(j,\{i,j\})}{1-f(i,\{i,j\})}\left(f(i,S) - f(i,S\setminus\{j\})\right).$$

This equation shows that the given distributed shares for all subsets with at most $m-1$ agents and a choice of a distributed share $f(i,S)$ uniquely determine the distributed shares $f(j,S)$ for every other agent j of S. Moreover, the distributed shares $\{f(j,S)\}_{j\neq i}$ are strictly increasing with the choice of $f(i,S)$. Therefore, there can be only one choice of $f(i,S)$ that satisfied our budget-balanced constraint $\sum_{j\in S} f(j,S) = 1$. This completes the proof. ■

Step #3: Viable Protocols are Weighted Shapley Values: The next lemma completes our proof by establishing an equivalence between the weighted Shapley value and protocols that are scalable, budget-balanced, positive, and guarantee the existence of an equilibrium in any game $G \in \mathcal{G}$.

Lemma 6.6: Let f be a protocol that is scalable, budget-balanced, positive, and guarantees the existence of an equilibrium in any game $G \in \mathcal{G}$. Then f must be a weighted Shapley value.

Proof: For the weighted Shapley values, it is straightforward to show that for a given set of weights $\{\omega_i\}$ and any agent $i \in N \setminus \{1\}$

$$f^{\mathrm{WSV}}(1,\{1,i\}) = \frac{\omega_i}{\omega_1+\omega_i}.$$

Define $\omega_1 = 1$. For each agent $i \in \{2, ..., n\}$, define ω_i be the solution to the following equation

$$\frac{\omega_i}{1+\omega_i} = f(1, \{i, 1\}).$$

Consider the weighted Shapley value protocol induced by the defined weight vector $\{\omega_i\}$. By definition, the protocols f and f^{WSV} have identical pairwise shares for agent 1. Since both f and f^{WSV} are scalable, positive, budget-balanced, and guarantee the existence of an equilibrium in any game $G \in \mathcal{G}$, then by Lemma 6.5 they are identical protocols. Therefore, the protocol f represents a weighted Shapley value. This completes the proof. ■

As mentioned previously, positivity is unnecessary for utility design in distributed engineering systems. We can remove positivity by showing an equivalence between a non-positive protocol and a concatenation of positive protocols as demonstrated in [39]. For brevity, we direct the readers to [39] for the concatenation arguments which are algebraic and directly applicable as we transition from cost minimization to welfare maximization resource allocation problems. This completes the proof of Theorem 3.1. □

B. Proof of Theorem 3.2

Consider a resource allocation problem with agent set $N = \{1, ..., n\}$, resource set $\mathcal{R} = \{r_0, r_1, ..., r_{n-1}\}$, and submodular welfare functions of the form

$$W_r(S) = \begin{cases} 0 & S = \emptyset \\ v_r & S \neq \emptyset \end{cases}$$

for each $r \in \mathcal{R}$. Define the class of distributed welfare games $\mathcal{G}$ by extending the above resource allocation problem to all possible action sets $\mathcal{A}_i \subseteq \mathcal{R}$ and resource values $v_r \geq 0$. Note that this class of games is a subset of the more general class of distributed welfare games with submodular welfare functions. Let G_1 represent a specific game instance with resource values $v_{r_0} = ... = r_{r_{n-1}} = 1$ and agent action sets $\mathcal{A}_1 = ... = \mathcal{A}_n = \{r_0\}$. This structure means that all agents $i \in N$ have only one actions and that is to select resource r_0. Without loss of generalities, suppose $f_{r_0}(1, N) \geq f_{r_0}(2, N) \geq ... \geq f_{r_0}(n, N)$. Since $f_{r_0}(\cdot)$ is budget-balanced, we have that $f_{r_0}(n, N) \leq 1/n$.

Let G_2 represent an alternative game instance with resource values $v_{r_0} = 1$ and $v_{r_i} = f_{r_0}(i, N) - \epsilon$ for each $i \in \{1, ..., n-1\}$ and for $\epsilon > 0$. Define the agents' actions sets as

$\mathcal{A}_i = \{r_0, r_i\}$ for each $i \in \{1, ..., n-1\}$ and $\mathcal{A}_n = \{r_0\}$ meaning that each agent $i \in \{1, ..., n-1\}$ is capable of selecting an alternative action while agent n is fixed at resource r_0. For any $\epsilon > 0$, the unique equilibrium is the profile $(r_0, ..., r_0)$ which yields a total welfare of 1. This is true since the weighted Shapley value protocol is monotonic which implies that for any subset $S \subseteq N$ and any agent $i \in S$, $f_{r_0}(i, S) \geq f_{r_0}(i, N) > f_{r_0}(i, N) - \epsilon$. The optimal allocation is $(r_1, r_2, ..., r_{n-1}, r_0)$, i.e, each agent $i \in \{1, ..., n-1\}$ selects resource r_i while agent n selects resource r_0. The optimal allocation yields a total welfare of

$$\begin{aligned} W(a^{\mathrm{opt}}) &= 1 + \sum_{i=1}^{n-1} (f_{r_0}(i, N) - \epsilon) \\ &= 2 - (n-1)\epsilon - f(n, N) \\ &\geq \frac{2n-1}{n} - (n-1)\epsilon. \end{aligned}$$

Since n and ϵ are arbitrary, this gives us a price of stability $\leq 1/2$.

From [2], we know that the weighted Shapley value guarantees a price of anarchy of $1/2$ since the welfare functions are submodular. By definition the price of stability is greater than or equal to the price of anarchy, hence we get that the price of stability equals $1/2$. This completes the proof. □