Efficiency and Revenue in Certain Nash Equilibria of Keyword Auctions

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Outline

• Model for keyword auctions.
• Efficiency in pure-strategy Nash equilibrium.
  • Necessary conditions for equilibrium.
  • Worst-case bound on efficiency.
• Revenue in symmetric equilibrium.
  • General case.
  • Restricted family of weights.
  • Efficiency and relevance considerations.
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Model

• $K$ positions, $N$ bidders.
• The click-through rate of bidder $s$ in positions $t$ is $e_s x_t$, i.e. separable into
  1. advertiser effect (or relevance) $e_s$
  2. position effect $x_t (x_1 > x_2 > \ldots > x_K)$.
• Bidder $s$ has per-click value of $v_s$.
• If bidder $s$ obtains slot $t$ at price of $p$ per click, utility is

$$e_s x_t (v_s - p),$$

i.e. quasi-linear.
Auction Rules

- The auctioneer assigns a weight $w_s$ to each bidder $s$ [Aggarwal et al. ’06].
- Bidders submit bids (reported values) $b_s$.
- Bidders are ranked in order of decreasing score $w_s b_s$.
- Bidder $s$ pays per click the lowest bid necessary to maintain its position:

\[ w_s b_s \geq w_{s+1} b_{s+1} \Rightarrow b_s \geq \frac{w_{s+1}}{w_s} b_{s+1} \]

- “Yahoo model”: $w_s = 1$.
- “Google model”: $w_s = e_s$. 
Efficient Ranking

• A bidder’s true score is $r_s = w_s v_s$.

• An allocation of slots to bidders $\sigma : K \rightarrow N$ maximizes the objective

$$\sum_t x_t w_{\sigma(t)} v_{\sigma(t)}$$

if and only if bidders are ranked in decreasing order of true score.

• Follows easily from the fact that $x_1 > x_2 \ldots > x_K$.

• If $w_s = e_s$, the objective is the total value.
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Pure-Strategy Nash Equilibrium

- Ad-hoc justification:
  bidders constantly update their bids until they find their current position is preferred to any other, given others’ bids.

- An allocation and vector of bids constitute a pure-strategy Nash equilibrium if, for each slot \( s \),

\[
e_s x_s (v_s - \frac{w_{s+1}}{w_s} b_{s+1}) \geq e_x x_t (v_s - \frac{w_{t+1}}{w_s} b_{t+1}) \quad (t > s)
\]

\[
\Leftrightarrow x_s (w_s v_s - w_{s+1} b_{s+1}) \geq x_t (w_s v_s - w_{t+1} b_{t+1}) \quad (t > s)
\]

and

\[
x_s (w_s v_s - w_{s+1} b_{s+1}) \geq x_t (w_s v_s - w_t b_t) \quad (t < s)
\]
Partial Characterization

• Can have a multiplicity of Nash equilibria, in terms of both bid vectors and allocations.

• What allocations can arise in Nash equilibrium?

**Lemma 1** An allocation can arise in pure-strategy Nash equilibrium only if

\[
    r_i \geq \frac{x_i}{x_{i+1}} \frac{x_{i+1} - x_j}{x_i - x_j} r_j
\]

for \( 1 \leq i \leq K - 2 \) and \( j \geq i + 2 \).

• Proof sketch: Farkas lemma.

• Complete characterization known for the case where \( N = 3 \) [Börgers et al., '07].
Bound on Efficiency

Example: 5 bidders, number them such that $r_1 \geq r_2 \geq \ldots \geq r_5$.

$$\frac{x_1 r_3 + x_2 r_2 + x_3 r_1 + x_4 r_5 + x_5 r_4}{x_1 r_1 + x_2 r_2 + x_3 r_3 + x_4 r_4 + x_5 r_5} \geq \min \left\{ \frac{x_1 r_3}{x_1 r_1}, \frac{x_2 r_2}{x_2 r_2}, \frac{x_3 r_1}{x_3 r_3}, \frac{x_4 r_5}{x_5 r_5}, \frac{x_5 r_4}{x_4 r_4} \right\}$$

$$= \min \left\{ \frac{x_1 r_3}{x_1 r_1}, \frac{x_5 r_4}{x_4 r_4} \right\}$$

$$\geq \min \left\{ \frac{x_1 x_2 - x_3}{x_2 x_1 - x_3}, \frac{x_5}{x_4} \right\}$$

In general, we can give a lower bound of

$$L = \min_{i=1,\ldots,N-1} \min \left\{ \frac{x_i}{x_{i+1}}, \frac{x_{i+1} - x_{i+2}}{x_i}, \frac{x_{i+1}}{x_i} \right\}$$
Exponential Decay

• For the exponential decay model, $x_t \sim 1/\delta^t$ for $\delta > 1$, we have

$$L = \min \left\{ \frac{\delta}{1+\delta}, \frac{1}{\delta} \right\}$$

• $\delta = 1.428$ [Feng et al., ’06], $L = 0.6$.
• $\delta = 1.5$ [Börgers et al., ’07], $L = 0.59$.

• My own estimates on a keyword, without exponential decay, $L = 0.07$...

• Need to pay particular attention to ordering at the top, and at breaks in the ad display.
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Symmetric Equilibrium

- To analyze revenue, assume bidders are playing a “locally envy-free equilibrium” [Edelman et al. ’06] “symmetric equilibrium” [Varian ’06].

- An allocation and vector of bids constitute a symmetric equilibrium if, for slots $s, t$,

$$e_s x_s (v_s - \frac{w_{s+1}}{w_s} b_{s+1}) \geq e_s x_t (v_s - \frac{w_{t+1}}{w_s} b_{t+1})$$

$$\Leftrightarrow x_s (w_s v_s - w_{s+1} b_{s+1}) \geq x_t (w_s v_s - w_{t+1} b_{t+1})$$
Properties

• Implies that complementary slackness conditions for the assignment problem hold.

• In symmetric equilibrium, bids are such that bidders are ranked in order of decreasing true score.
  - \( w_s v_s \geq w_t v_t \iff w_s b_s \geq w_t b_t \).
  - Maximizes objective \( \sum_t x_t w_{\sigma(t)} v_{\sigma(t)} \).
  - If \( w_s = e_s \), symmetric equilibrium is efficient.

• Set of symmetric equilibrium bids forms a lattice [Shapley and Shubik, ’72].
  - There exist minimum and maximum bid vectors.
  - Select the minimum element, to optimize a lower bound on revenue.
  - When \( w_s = e_s \), minimum element gives Vickrey payments [Leonard, ’83].
Payments in Symmetric Equilibrium

Let

\[
y_{st}(e,v) = \begin{cases} 
1 & \text{if bidder } s \text{ gets slot } t \\
0 & \text{otherwise}
\end{cases}
\]

The total payment of bidder \( s \) in minimum symmetric equilibrium is

\[
\sum_{t=s}^{K} \frac{w_{t+1}}{w_s} e_s(x_t - x_{t+1})v_{t+1}
\]

\[
= e_s \sum_{t=1}^{n} x_t \left[ v_s y_{st}(e,v) - \int_0^{v_s} y_{st}(e,\tau,v_{-s}) \, d\tau \right].
\]

Very similar to [Myerson ’81]’s analysis of the single-item case.
General Problem Formulation

$$\max_{w} \int_{[0,1]^N} \int_{[0,\infty]^N} \sum_{s=1}^{N} \sum_{t=1}^{K} x_t e_s \psi(e_s, v_s) y_{st}(e, v) f(e, v) \, dv \, de$$

s.t.

$$\sum_{t} y_{st}(e, v) \leq 1 \quad \forall s, e, v$$

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Would like to rank bidders according to their “virtual scores” $e_s \psi(e_s, v_s)$.

$$\psi(e_s, v_s) = v_s - \frac{1 - F(v_s | e_s)}{f(v_s | e_s)}$$

But we are restricted to ranking according to scores of the form

$$g(e_s)v_s + h(e_s).$$
Restricted Family of Weights

We restrict our attention to weights

\[ w_s = e^q_s \]

where \( q \in (-\infty, +\infty) \).

- Yahoo model: \( q = 0 \).
- Google model: \( q = 1 \).

Recall the equilibrium payment:

\[
\sum_{t=s}^{K} \left( \frac{e_{t+1}}{e_s} \right)^q e_s (x_t - x_{t+1}) v_{t+1}
\]
Efficiency and Relevance

Total relevance is \( \sum_{t=1}^{K} e_t x_t \).

**Proposition 1** Total relevance is non-decreasing in \( q \).

**Intuition:** the impact of relevance on the score increases relative to the bid as \( q \) increases.

Total value (efficiency) is \( \sum_{t=1}^{K} e_t x_t v_t \).

**Proposition 2** Total value is non-decreasing in \( q \) for \( q \leq 1 \) and non-increasing in \( q \) for \( q \geq 1 \).

**Intuition:** at \( q = 1 \), efficiency is maximized in equilibrium.
Simulations

- Obtained bid and click-through rate data for advertisers bidding on a high-volume keyword in summer of 2006.
- Bidder values estimated by deriving bounds according to symmetric equilibrium [Varian ’06].
- Marginal distributions:
  - Value: Lognormal, $\mu = 0.35$ and $\sigma = 0.71$.
  - Relevance: Beta, $a = 2.71$ and $b = 25.43$.
- Spearman correlation between value and relevance was in $[0.36, 0.55]$ over a month.
- Modeled dependence between value and relevance with a Gaussian copula.
Revenue Effect of Correlation

![Graph showing the revenue effect of correlation with respect to q. The graph has multiple curves, each representing different values of q: -1, -0.5, 0, 0.5, and 1. The x-axis represents q ranging from -2 to 2, and the y-axis represents R(q) ranging from 0 to 7.]
Correlation of 0.4
Efficiency Bounds

![Graph showing the relationship between R(q) and q]
Relevance Lower Bound

![Graph showing the relation between R(q) and q. The graph has a peak at q around 0.5, with R(q) values ranging from 0.6 to 3.5. The x-axis represents q values ranging from -2 to 2, and the y-axis represents R(q) values ranging from 0.6 to 3.5.](image-url)
Bidding Credits

Suppose bidder $s$ only pays a fraction $c_s \in [0, 1]$ of the price he faces. The symmetric equilibrium constraints are

$$e_s x_s (v_s - c_s \frac{w_{s+1}}{w_s} b_{s+1}) \geq e_s x_t (v_s - c_s \frac{w_{t+1}}{w_s} b_{t+1})$$

$$\iff x_s \left( \frac{w_s}{c_s} v_s - w_{s+1} b_{s+1} \right) \geq x_t \left( \frac{w_s}{c_s} v_s - w_{t+1} b_{t+1} \right)$$

Letting $w'_s = \frac{w_s}{c_s}$ and $b'_s = c_s b_s$, we get

$$x_s (w'_s v_s - w'_{s+1} b'_{s+1}) \geq x_t (w'_s v_s - w'_{t+1} b'_{t+1})$$

Back to the original symmetric equilibrium inequalities!

- Revenue will be the same as if we had used weights $w'$.
- To go from Google-model revenue to Yahoo-model revenue, set credits to $c_s = e_s$. 
Summary

• With correlation of $0.4$, $q = 0$ is optimal, yielding 25% more revenue than $q = 1$.

• Imposing a bound of 5% loss in efficiency and relevance from baseline of $q = 1$, $q = 0.6$ is optimal with 11% improvement in revenue.

• Optimal reserve score is 0.2—it gives 8% increase in revenue, but 13% efficiency loss and 26% relevance loss.

• In theory, same effect could be achieved with bidding credits.
Conclusions

• Pure-strategy Nash equilibria of keyword auctions should be quite efficient.

• Open question as to whether “swaps” are a problem.

• Optimal keyword auction design problem can be formulated as a mathematical program.

• Open question as to how to solve the program for general weighting schemes.

• Changing exponent $q$ on advertiser effect can improve revenue, depending on correlation between value and relevance.

• Simulations suggest this approach can be more effective than using a reserve score.