The Price of Anarchy Revisited

• Objective function: Average latency

• For two-link, two node network with arbitrary latency functions, the price of anarchy, $\rho$, was unbounded

• For linear latency functions, $\rho \leq 4/3$ independent of the network topology

• The Braess ratio $\beta$ is also bounded by $4/3$ for linear latency functions and is bounded by $n/2$ for arbitrary latencies

• Other possible objectives?
The Price of Anarchy Revisited

- Fairness: Objective is maximum latency

- The price of anarchy in this case is $n - 1$ for a single commodity and arbitrary latency functions

- Conjecture: The price of anarchy and the Braess ratio are dependent only on the topology, and not the edge latency functions

- Conjecture: The $n - 1$ bound is still true for multicommodity networks

- Note that for the average latency objective, there is no separation in the behavior of $\rho$ for single and multicommodity networks
The Braess Ratio in Multicommodity networks

- Perhaps surprisingly, Braess’s paradox can be much more severe in multicommodity networks.

- There is a phase transition of sorts: While $\beta$ is polynomial in the single commodity instances, it can be exponential with just two commodities.

- Example: A family of networks that is closely related to the Fibonacci numbers. Recall that the $p^{th}$ Fibonacci number is approximately equal to $c \cdot \phi^p$, with $c \approx 0.4772$ and $\phi$ the golden ratio.
Model

- $G = (V, E)$, $K$ source-destination pairs, traffic rate $r_i$
- $P_i = s_i - t_i$ paths in $G$. $\mathcal{P} = \bigcup_{i=1}^{k} P_i$
- Feasible flow $f_p$ routes all traffic. $f_e = \sum_{p} f_p$
- Latency: $l_p(f) = \sum_{e \in P} l_e(f_e)$, where $l(e)$ is the edge latency function
- Objective: $M(f) = \max_{p \in \mathcal{P}: f_p > 0} l_p(f)$. Let $L_i$ be the common latency of the $i^{th}$ commodity
Theorem (Lin, Roughgarden, Tardos, Walkover 2005)

• There is an infinite family \( \{(G^p, r^p, l^p)\}_{p=1}^{\infty} \) with the following properties

• \((G^p, r^p, l^p)\) has two commodities and \(O(p)\) vertices and edges;

• For \(p\) odd, \(L_1(G^p, r^p, l^p) = F_{p-1} + 1\) and \(L_2(G^p, r^p, l^p) = F_p\);

• For all \(p\), there is a subgraph \(H^p\) of \(G^p\) with one less edge than \(G^p\) that satisfies \(L_1(G^p, r^p, l^p) = 1\) and \(L_2(G^p, r^p, l^p) = 0\)
Prevalence of Braess’s Paradox

- Is Braess’s paradox just a mere theoretical curiosity?

- Hardness of finding Braess edges (Roughgarden 2002)

- Theoretical investigations of random graphs show that Braess’s paradox does occur with high probability in random graphs as the number of vertices increases

- What is random here?
Model (Valiant and Roughgarden 2006)

- G is the common Erdös-Renyi random graph model

- Each possible edge is present with probability $p$

- Latency functions are random too, in the following sense: Let the functions be affine (i.e. $ax + b$), where $a$ and $b$ are chosen from some fixed distributions $A$ and $B$, respectively

- The results are also true for the $1/x$ model: For every edge, the latency is either $x$ (with probability $p$), or 1 (with probability $1 - p$)
Theorem Valiant and Roughgarden (2006)

Let $A$ and $B$ be reasonable distributions. There is a constant $\rho = \rho(A, B) > 1$ such that, with high probability, a random network $(G, l)$ admits a choice of traffic rate $r$ such that the Braess ratio of the instance $(G, r, l)$ is at least $\rho$.