CS/SS 241a **Assignment 3** **Guru: Jason Marden**
Assigned: 1/31/08 Due: 2/14/08 2:30pm

We encourage you to discuss these problems with others, but you need to write up the actual homework alone. At the top of your homework sheet, please list all the people with whom you discussed each problem. Crediting help from other classmates will not take away any credit from you, and will prevent us from assuming cheating if your answers look similar.

Please limit yourself to the course textbook and your course notes when working on the homework. Do not search the web if you become stuck – Google is all-knowing.

1 Equilibrium Comparisons

1.1 Nash Equilibrium

The most well known form of an equilibrium is the Nash equilibrium.

Definition 1.1 (Pure Nash Equilibrium) *An action profile $a^* \in \mathcal{A}$ is called a pure Nash equilibrium if for all players $\mathcal{P}_i \in \mathcal{P}$,*

$$U_i(a_i^*, a_{-i}^*) = \max_{a_i \in \mathcal{A}_i} U_i(a_i, a_{-i}^*). \tag{1}$$

Furthermore, if the above condition is satisfied with a unique maximizer for every player $\mathcal{P}_i \in \mathcal{P}$, then a^* is called a strict Nash equilibrium.

A Nash equilibrium represents a scenario for which no player has an incentive to unilaterally deviate.

The concept of Nash equilibrium also extends to mixed strategy spaces. Let the *strategy* of player $\mathcal{P}_i$ be defined as $p_i \in \Delta(\mathcal{A}_i)$, where $\Delta(\mathcal{A}_i)$ is the set of probability distributions over the finite set of actions $\mathcal{A}_i$. We will adopt the convention that $p_i^{a_i}$ represents the probability that player $\mathcal{P}_i$ will select action a_i and $\sum_{a_i \in \mathcal{A}_i} p_i^{a_i} = 1$. If all players $\mathcal{P}_i \in \mathcal{P}$ play independently according to their personal strategy $p_i \in \Delta(\mathcal{A}_i)$, then the expected utility of player $\mathcal{P}_i$ for strategy p_i is defined as

$$U_i(p_i, p_{-i}) = \sum_{a \in \mathcal{A}} U_i(a) p_1^{a_1} p_2^{a_2} \dots p_n^{a_n},$$

where $p_{-i} = \{p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n\}$ denotes the collection of strategies of players other than player $\mathcal{P}_i$.

Definition 1.2 (Nash Equilibrium) *A strategy profile $p^* = \{p_1^*, \dots, p_n^*\}$ is called a Nash equilibrium if for all players $\mathcal{P}_i \in \mathcal{P}$,*

$$U_i(p_i^*, p_{-i}^*) = \max_{p_i \in \Delta(\mathcal{A}_i)} U_i(p_i, p_{-i}^*). \tag{2}$$

1.2 Correlated Equilibrium

In this section we will define a broader class of equilibria for which there may be correlations among the players. To that end, let $z \in \Delta(\mathcal{A})$ denote a probability distribution over the set of joint actions $\mathcal{A}$. We will adopt the convention that z^a is the probability of the joint action a and $\sum_{a \in \mathcal{A}} z^a = 1$. In the special case that all players $\mathcal{P}_i \in \mathcal{P}$ play independently according to their personal strategy $p_i \in \Delta(\mathcal{A}_i)$, as was the case in the definition of the Nash equilibrium, then

$$z^a = p_1^{a_1} p_2^{a_2} \dots p_n^{a_n},$$

where $a = (a_1, a_2, \dots, a_n)$.

Definition 1.3 (Correlated Equilibrium) *The probability distribution z is a correlated equilibrium if for all players $\mathcal{P}_i \in \mathcal{P}$ and for all actions $a_i, a_i' \in \mathcal{A}_i$,*

$$\sum_{a_{-i} \in \mathcal{A}_{-i}} U_i(a_i, a_{-i}) z^{(a_i, a_{-i})} \geq \sum_{a_{-i} \in \mathcal{A}_{-i}} U_i(a_i', a_{-i}) z^{(a_i, a_{-i})}. \tag{3}$$

To motivate this definition consider the following scenario. First, a joint action $a \in \mathcal{A}$ is randomly drawn according to the probability distribution $z \in \Delta(\mathcal{A})$. Next, each player is informed of only his particular action a_i, but not the actions of the other players. Finally, each player is given the opportunity to change his action. The condition for correlated equilibrium in (3) states that each player $\mathcal{P}_i$'s conditional expected payoff for action a_i is at least as good as his conditional expected payoff for any other action $a_i' \neq a_i$. In other words, a probability distribution z is a correlated equilibrium if and only if no player would seek to change his action from the outcome, randomly drawn according to z, even after his part has been revealed.

1.3 Coarse Correlated Equilibrium

We will now introduce a third equilibrium notion called coarse correlated equilibrium. Before doing so, we will discuss marginal distributions. Given the joint distribution $z \in \Delta(\mathcal{A})$, the marginal distribution of all players other than player $\mathcal{P}_i$ is

$$z_{-i}^{a_{-i}} = \sum_{a_i' \in \mathcal{A}_i} z^{(a_i', a_{-i})}.$$

Note that z_{-i} is a well defined probability distribution in $\Delta(\mathcal{A}_{-i})$.

Definition 1.4 (Coarse Correlated Equilibrium) *The probability distribution z is a coarse correlated equilibrium if for all players $\mathcal{P}_i \in \mathcal{P}$ and for all actions $a_i' \in \mathcal{A}_i$,*

$$\sum_{a \in \mathcal{A}} U_i(a) z^a \geq \sum_{a_{-i} \in \mathcal{A}_{-i}} U_i(a_i', a_{-i}) z_{-i}^{a_{-i}}. \tag{4}$$

To motivate this definition, consider the following scenario which differs slightly from the correlated equilibrium scenario. Before the joint action a is drawn, each player $\mathcal{P}_i$ is given the opportunity to opt out, in which case the player can select any action $a_i \in \mathcal{A}_i$ that he wishes. If the player does not opt out, he commits himself to playing his part of the action-tuple a randomly drawn according to the distribution z. In words, a distribution z is a coarse correlated equilibrium if under this scenario no player would choose to opt out given that all other players opt to stay in.

(a) Consider the following two player game. Characterize the set of Nash equilibria, Correlated equilibria, and Coarse Correlated equilibria. Extra credit if you also do Endogenous Correlated Equilibrium.

	Red	Yellow	Blue
Red	$1,0$	0,0	$0,1$
Yellow	$0,1$	1,0	$0,0$
Blue	$0,0$	0,1	$1,0$

(b) Derive the relationship between Nash equilibria, correlated equilibria, and coarse correlated equilibria and illustrate the results using a Venn diagram. Prove all cases! Give examples. For example, give an example of a coarse correlated equilibrium that is not a Nash equilibrium.

(c) There is also the notion of Endogenous Correlated Equilibrium. (You can use google to find the precise definition). Derive the relationship between Endogenous Correlated Equilibria and Nash, Correlated and Coarse Correlated. Illustrate your finding on the Venn diagram.

(d) Discuss the relationship between the four equilibria concepts and the notion of regret, i.e., do the equilibria exhibit no-regret, no-conditional regret, equal regret?

(e) Consider two joint distributions. Let z^{ne} be a joint distribution of a mixed Nash equilibrium and z^{cce} be the distribution of a Coarse Correlated equilibrium. Can you make any definitive statements regarding the expected payoffs to players at z^{ne} or z^{cce}. For example, is $U_i(z^{ne}) \geq U_i(z^{cce})$? If no statement can be made, provide an example.

2 Classes of Games

In class, we identified four classes of games: identical interest, potential, congestion, and weakly acyclic games.

1. Consider the following two player game.

	a_2	b_2
a_1	$0, 1$	$-3, 2$
b_1	$2, 4$	$-1, 6$

Is this a potential game? If so, what is the potential function. Simulate the following learning dynamics (using Matlab) on this game from a variety of initial conditions. Document your results. For example, plot the evolution of the joint action profile and evolution of empirical frequency. Do your findings match the theory?

(a) Fictitious Play

(b) Regret Matching

(c) Regret Matching with Fading Memory and Inertia – “Regret Based Dynamics: Convergence in Weakly Acyclic Games,” J.R. Marden, G. Arslan and J.S. Shamma

(d) Joint Strategy Fictitious Play with Inertia – “Joint Strategy Fictitious Play with Inertia for Potential Games,” J.R. Marden, G. Arslan and J.S. Shamma

(e) (Extra Credit) Fading Memory Joint Strategy Fictitious Play with Inertia – “Joint Strategy Fictitious Play with Inertia for Potential Games,” J.R. Marden, G. Arslan and J.S. Shamma

(f) (Extra Credit) Simple Experimentation Dynamics – “Payoff Based Dynamics for Multi-Player Weakly Acyclic Games,” J.R. Marden, H. P. Young, G. Arslan and J.S. Shamma

3 Weakly Acyclic Games

Consider the following game:

	a_2	b_2	c_2
a_1	$-100, 0$	0.1,0	$1, 1$
b_1	$1, 0$	0,1	$0, -100$
c_1	$0, 1$	1,0	$0, -100$

(a) Prove that this game is weakly acyclic using the original definition and the potential function condition as derived in class.

(b) (Matlab) Simulate regret matching on this two player game from a variety of initial conditions. Document your results.

(c) (Matlab) Simulate regret matching with fading memory and inertia (see paper cited in previous question) on this two player game from a variety of initial conditions. Document your results.

4 Congestion Games – Simulations (Matlab)

Congestion games are a specific class of games in which player utility functions have a special structure.

In order to define a congestion game, we must specify the action set, $\mathcal{A}_i$, and utility function, $U_i(\cdot)$, of each player. Towards this end, let $\mathcal{R}$ denote a finite set of "resources". For each resource $r \in \mathcal{R}$, there is an associated "congestion function"

$$c_r : \{0, 1, 2, ...\} \to \mathbb{R}$$

that reflects the cost of using the resource as a function of the number of players using that resource.

The action set, $\mathcal{A}_i$, of each player, $\mathcal{P}_i$, is defined as the set of resources available to player $\mathcal{P}_i$, i.e.,

$$\mathcal{A}_i \subset 2^{\mathcal{R}},$$

where $2^{\mathcal{R}}$ denotes the set of subsets of $\mathcal{R}$. Accordingly, an action, $a_i \in \mathcal{A}_i$, reflects a selection of (multiple) resources, $a_i \subset \mathcal{R}$. A player is "using" resource r if $r \in a_i$. For an action profile $a \in \mathcal{A}$, let $\sigma_r(a)$ denote the total number of players using resource r, i.e., $|\{i : r \in a_i\}|$. In a congestion game, the utility of player $\mathcal{P}_i$ using resources indicated by a_i depends only on the total number of players using the same resources. More precisely, the utility of player $\mathcal{P}_i$ is defined as

$$U_i(a) = \sum_{r \in a_i} c_r(\sigma_r(a)). \tag{5}$$

Create an arbitrary congestion game of any topology with at least 10 resources. (You design the cost functions – maybe linear or polynomial in the number of users.) Suppose there are 1000 players who have the ability to select any possible resource. We know that a congestion game is a potential game and there are learning algorithms that guarantee convergence to a pure Nash equilibrium in any potential game. Simulate the following learning algorithms on your congestion game – document your findings with observations and graphs for each dynamics if applicable. For example, plot evolution of joint action and evolution of empirical frequency. Do your findings match the theory?

(a) Fictitious Play

(b) Regret Matching

(c) Regret Matching with Fading Memory and Inertia – "Regret Based Dynamics: Convergence in Weakly Acyclic Games," J.R. Marden, G. Arslan and J.S. Shamma

(d) Joint Strategy Fictitious Play with Inertia – "Joint Strategy Fictitious Play with Inertia for Potential Games," J.R. Marden, G. Arslan and J.S. Shamma

(e) (Extra Credit) Fading Memory Joint Strategy Fictitious Play with Inertia – "Joint Strategy Fictitious Play with Inertia for Potential Games," J.R. Marden, G. Arslan and J.S. Shamma

(f) (Extra Credit) Simple Experimentation Dynamics – "Payoff Based Dynamics for Multi-Player Weakly Acyclic Games," J.R. Marden, H. P. Young, G. Arslan and J.S. Shamma

5 Congestion Games – Theory

Consider the definition of a congestion game as described above. Suppose each player's action set is $\mathcal{A}_i = \mathcal{R}$ where $|a_i| = 1$ for any $a_i \in \mathcal{A}$. This means that each player can select any single resource. Suppose cost functions are increasing in the number of users. Does a pure Nash equilibrium exist? Is it unique?