CS/SS 241a | **Assignment 2** | **Guru: Adam Wierman**

Assigned: 1/17/08 | Due: 1/29/08 2:30pm

We encourage you to discuss these problems with others, but you need to write up the actual homework alone. At the top of your homework sheet, please list all the people with whom you discussed each problem. Crediting help from other classmates will not take away any credit from you, and will prevent us from assuming cheating if your answers look similar.

Please limit yourself to the course textbook and your course notes when working on the homework. Do not search the web if you become stuck – Google is all-knowing.

1 How many players does it take to maximize anarchy? [35 points]

In this problem, we return to the atomic selfish routing game.

(a) In class we saw an example with 4 players and linear cost functions where the Price of Anarchy was 5/2 (the AAE example). I mentioned that the worst case PoA for atomic games with linear cost functions is ϕ^2, where ϕ is the golden ratio. Provide an example that attains this PoA.

(b) Is it possible to a devise an example using only 3 players and linear costs that attains a price of anarchy of ϕ^2?

(c) What is the largest price of anarchy in an atomic instance with linear costs and 2 players?

2 Generalizing Pigou's bound [35 points]

In class we defined Pigou's bound as

$$\alpha(C) = \sup_{c \in C} \sup_{x,r \geq 0} \frac{rc(r)}{xc(x) + (r-x)c(r)}$$

Recall, that we proved that Pigou's bound is a lower bound on the price of anarchy when C includes all of the constant cost functions. That is, for such classes C, the price of anarchy in nonatomic instances can be arbitrarily close to $\alpha(C)$. We depended on the existence of constant cost functions in our proof, but this problem will illustrate that this was not necessary.

(a) Prove that $\alpha(C)$ is a lower bound on the price of anarchy when C is *diverse*, i.e., for every positive scalar $\gamma > 0$ there is a cost function $c \in C$ satisfying $c(0) = \gamma$.

(b) Prove that $\alpha(C)$ is a lower bound on the price of anarchy when C is *inhomogeneous*, i.e., there is at least one cost function $c \in C$ satisfying $c(0) > 0$.

Hint: Simulate a 2-node Pigou-like example using a more complex network.

3 Applying Pigou's bound [30 points]

Not only did we prove that the Pigou bound $\alpha(C)$ defined above is a lower bound on the price of anarchy, we also proved that it is an upper bound. In this problem, you will apply the Pigou bound to determine the price of anarchy for a few important classes of cost functions.

(a) Evaluate the Pigou bound to prove that the price of anarchy for nonnegative, nondecreasing, concave cost functions is $4/3$.
Hint: Use the fact that $c(\lambda x) \geq \lambda c(x)$ for $\lambda \in [0, 1]$.

(b) Evaluate the Pigou bound to prove that the price of anarchy for degree-p polynomial cost functions with non-negative coefficients is

$$\frac{1}{1 - p(p+1)^{-(p+1)/p}}.$$

Hint: Use the fact that $c(\lambda x) \geq \lambda^p c(x)$ for $\lambda \in [0, 1]$.

(c) In class we proved a bound of p for the PoA of non-negative polynomial cost functions. Evaluate the growth rate of the price of anarchy of non-negative degree p polynomial cost functions that you proved in (b) as $p \to \infty$. Is this growth rate an improvement over the linear growth rate of the bound in class?