CS 38 Introduction to Algorithms	Spring 2014

Problem Set 5

Out: May 15 — *Due:* **May 22**

Note: the due date has been shifted due to Ditch Day.

Reminder: you are encouraged to work in groups of two or three; however you must turn in your own write-up and note with whom you worked. You may consult the course notes and the optional text (CLRS). The full honor code guidelines can be found in the course syllabus.

Please attempt all problems. **To facilitate grading, please turn in each problem on a separate sheet of paper and put your name on each sheet. Do not staple the separate sheets.**

1. Given a flow network with edge capacities $c(e)$, consider the following variants of the maximum flow problem. For each, formulate them as a linear program.
   (a) We are interested in vertex-limited flow: for each vertex v there is a limit $g(v)$, and we want the flow entering v plus the flow exiting v to not exceed $g(v)$. We are interested in the maximum s-t flow subject to these additional constraints.
   (b) There is a distinguished edge e^* which is "fragile". We are interested in the maximum s-t flow that minimizes the flow on edge e^*. That is, the flow value should be maximum, but among all such maximum flows, we want the one with the smallest flow across edge e^*.

2. Total unimodularity. A matrix is said to be totally-unimodular if every square submatrix (obtained by restricting to a subset of the rows and subset of the columns) has determinant 0, 1, or -1. The polytope $\{x : Ax = b, x \in R^n \geq 0\}$ has only integer vertices if A is totally unimodular and b is integer.
   (a) Formulate the Assignment Problem as a linear program in which you assume that the variables take on only integer values (this is called an integer linear program). Show that the constraint matrix is totally unimodular, and hence solving this linear program gives a legitimate solution to the Assignment Problem.
   (b) Suppose a car company has m storage lots and n retail showrooms, and it needs to ship cars from storage lots to showrooms. Each storage lot has a supply a_i and each showroom has a demand b_i. We are also given costs $c_{i,j}$ between every storage lot and showroom. Shipping x units from storage lot i to showroom j costs $c_{i,j}x$. You want to determine an optimal shipment scheme subject to the supply and demand constraints, that minimizes total cost. Formulate this as a linear program and then show that the constraint matrix is totally unimodular (so you are never asked to ship 2/3 of a car!)

3. Zero-sum games and Nash Equilibria. Consider two players called the "row player" and the "column player" who each have n possible moves to chose from. An $n \times n$ matrix M specifies

the payoffs as follows: if the row player chooses move i and the column player chooses move j, then the payoff to the row-player is $M[i,j]$ and the payoff to the column player is $-M[i,j]$ (hence the term "zero sum"). A *mixed strategy* is one in which the player chooses from some distribution of moves, that is they choose move i with probability p_i (and we have $p_i \geq 0$ and $\sum_i p_i = 1$). A pair of mixed strategies p and q, for the row-player and column-player gives rise to an expected payoff to the row-player of $\sum_{i,j} p_i q_j M[i,j]$ and an expected payoff to the column-player of $-\sum_{i,j} p_i q_j M[i,j]$. Such a pair is said to be a Nash equilibrium if no player has any incentive to unilaterally deviate from their current mixed strategy. That is,

(a) for any $p' \neq p$, we have $\sum_{i,j} p'_i q_j M[i,j] \leq \sum_{i,j} p_i q_j M[i,j]$, and

(b) for any $q' \neq q$, we have $-\sum_{i,j} p_i q'_j M[i,j] \leq -\sum_{i,j} p_i q_j M[i,j]$.

An important question is how to efficiently find such an equilibrium.

(a) Suppose you are given the sets $S = \{i : p_i > 0\}$ and $T = \{j : q_j > 0\}$ for a pair p, q that represent a Nash equilibirum. Show how to find a Nash equilibrium in polynomial time using linear programming and this information.

(b) Give a $2^{O(2n)} \cdot \mathrm{poly}(n)$ time algorithm to find a Nash equilibrium given the payoff matrix M.