CS 21 Decidability and Tractability **Winter 2024**

Midterm

Out: January 31 *Due:* **February 7**

This is a midterm. You may consult only the course notes and the text (Sipser). *You may not collaborate.* The full honor code guidelines can be found in the course syllabus.

There are 5 problems on 2 pages. Please attempt all problems. **Please turn in your solutions via Gradescope, by 1pm on the due date.** Good luck!

1. Identify each of the following languages as either (i) regular, (ii) context-free but not regular, or (iii) not context-free. For each language, prove that your classification is correct, using the techniques we have developed in this course.

   (a) $L_1 = \{a^i b^j c^i d^j : i, j \geq 0\}$.

   (b) $L_2 = \{a^i b^j c^j d^i : i, j \geq 0\}$.

   (c) $L_3 = \{a^i b^j c^k : i = j = k \text{ or } i > 1000\}$.

2. Identify each of the following languages as either decidable or undecidable, and prove that your classification is correct, using the techniques we have developed in this course. Recall that for a context free grammar G, we denote by $L(G)$ the language it describes, and similarly for a regular expression E, we denote by $L(E)$ the language it describes.

   (a) $\text{CFL-IN-REG} = \{(G, E) : G \text{ is a CFG, } E \text{ is a regular expression, and } L(G) \subseteq L(E)\}$

   (b) $\text{REG-IN-CFL} = \{(E, G) : G \text{ is a CFG, } E \text{ is a regular expression, and } L(E) \subseteq L(G)\}$

   Hint: you may wish to use the fact that the intersection of a context free language and a regular language is context-free (Sipser problem 2.18).

3. Two (disjoint) languages L_1 and L_2 are called *recursively separable* if there is a decidable language D for which $L_1 \cap D = \emptyset$ and $L_2 \subseteq D$; they are *recursively inseparable* if no such decidable language D exists. Convince yourself that an undecidable language and its complement are recursively inseparable.

   Consider the following languages:

   $$L_1 = \{\langle M \rangle : M \text{ halts and accepts input } \langle M \rangle\}$$
   $$L_2 = \{\langle M \rangle : M \text{ halts and rejects input } \langle M \rangle\}$$

   Prove that L_1 and L_2 are recursively inseparable. Hint: your proof will probably involve supplying a Turing Machine its own description as input.

4. A *right-linear* CFG is a context-free grammar in which every production has the form

   - $A \to xB$, or
   - $A \to x$,

   where A and B are non-terminals, and x can be any string of terminals. A CFG is *linear* if productions of the form $A \to Bx$ are allowed in addition to the two types of productions in a right-linear CFG.

   (a) Prove that every language generated by a right-linear CFG is regular.

   (b) Prove that every regular language is generated by some right-linear CFG.

   (c) Give a linear CFG that generates the non-regular language over the alphabet $\Sigma = \{a, b\}$,

   $$L = \{w : w \text{ is a palindrome}\},$$

   and prove that your grammar indeed generates exactly L (i.e., prove that every string in L is generated by your grammar, and prove that every string generated by your grammar is in L).

5. Given a language L, define 3-IN-A-ROW$_L$ as follows:

   $$\begin{aligned}\text{3-IN-A-ROW}_L \quad = \quad & \{\#x_1\#x_2\#\cdots\#x_k\# : k \geq 0 \\ & \qquad \text{and for some } i,\ x_i \in L \text{ and } x_{i+1} \in L \text{ and } x_{i+2} \in L.\}\end{aligned}$$

   Prove that 3-IN-A-ROW$_L$ is RE if L is RE. Here the x_i are strings over L's alphabet, and # is a symbol that is not in L's alphabet.