CS21
Decidability and Tractability

Lecture 5
January 13, 2023

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Outline

- Non-regular languages: Pumping Lemma
- Pushdown Automata
- Context-Free Grammars and Languages

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Non-regular languages

Pumping Lemma: Let L be a regular language. There exists an integer p ("pumping length") for which every $w \in L$ with $|w| \geq p$ can be written as

$w = xyz$ such that

1. for every $i \geq 0$, $xy^iz \in L$, and
2. $|y| > 0$, and
3. $|xy| \leq p$.

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Non-regular languages

- Using the Pumping Lemma to prove L is not regular:
  - assume L is regular
  - then there exists a pumping length p
  - select a string $w \in L$ of length at least p
  - argue that for every way of writing $w = xyz$ that satisfies (2) and (3) of the Lemma, pumping on y yields a string not in L.
  - contradiction.

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Pumping Lemma Examples

- Theorem: $L = \{0^i1^j: i > j\}$ is not regular.
- Proof:
  - let p be the pumping length for L
  - choose $w = 0^{p+1}1^p$

$$w = \underbrace{000000000\ldots0}_{p+1}\underbrace{1111111\ldots1}_{p}$$

  - $w = xyz$, with $|y| > 0$ and $|xy| \leq p$.

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Pumping Lemma Examples

- 1 possibility:

$$w = \underbrace{0000}_{x}\underbrace{000}_{y}\underbrace{0\ldots0111111111\ldots1}_{z}$$

- pumping on y gives strings in the language (?)
- this seems like a problem…
- Lemma states that for every $i \geq 0$, $xy^iz \in L$
- xy^0z not in L. So L not regular.

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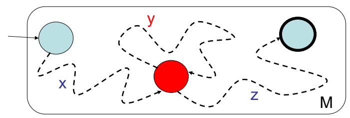
Proof of the Pumping Lemma

- Let M be a FA that recognizes L.
- Set p = number of states of M.
- Consider w ∈ L with |w| ≥ p. On input w, M must go through *at least* p+1 states. There must be a repeated state (among first p+1).



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FA Summary

- A "problem" is a language
- A "computation" receives an input and either accepts, rejects, or loops forever.
- A "computation" recognizes a language (it may also decide the language).
- Finite Automata perform simple computations that read the input from left to right and employ a finite memory.

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FA Summary

- The languages recognized by FA are the regular languages.
- The regular languages are closed under union, concatenation, and star.
- Nondeterministic Finite Automata may have several choices at each step.
- NFAs recognize exactly the same languages that FAs do.

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FA Summary

- Regular expressions are languages built up from the operations union, concatenation, and star.
- Regular expressions describe exactly the same languages that FAs (and NFAs) recognize.
- Some languages are not regular. This can be proved using the Pumping Lemma.

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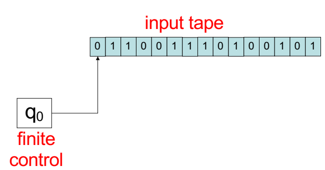
Machine view of FA



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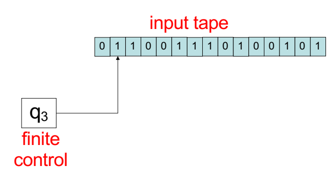
Machine view of FA



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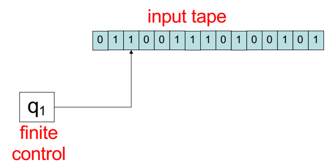
Machine view of FA



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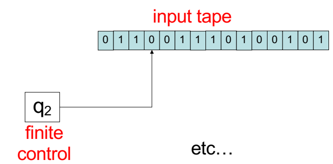
Machine view of FA



etc…

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A more powerful machine

- limitation of FA related to fact that they can only “remember” a bounded amount of information
- What is the simplest alteration that adds unbounded “memory” to our machine?
- Should be able to recognize, e.g., $\{0^n1^n : n \geq 0\}$

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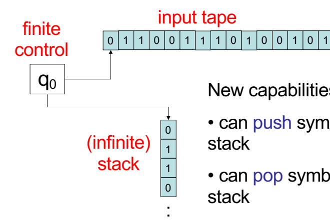
Pushdown Automata



New capabilities:

- can push symbol onto stack
- can pop symbol off of stack

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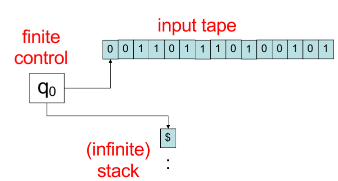
Pushdown Automata



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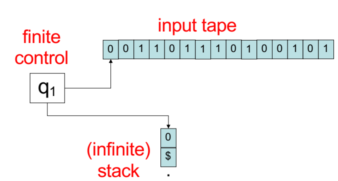
Pushdown Automata



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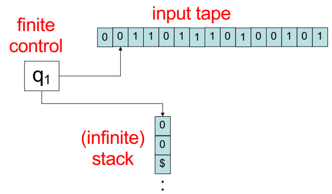
Pushdown Automata



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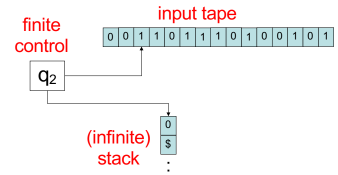
Pushdown Automata



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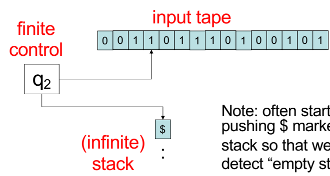
Pushdown Automata



Note: often start by pushing $ marker onto stack so that we can detect "empty stack"

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Pushdown Automata (PDA)

- We will define nondeterministic pushdown automata immediately
  - potentially several choices of "next step"
- Deterministic PDA defined later
  - weaker than NPDA
- Two ways to describe NPDA
  - diagram
  - formal definition

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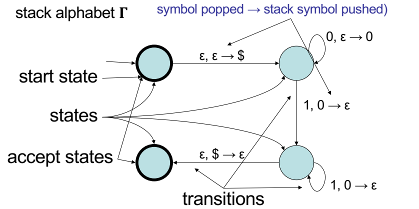
NPDA diagram

tape alphabet Σ

stack alphabet Γ

transition label: (tape symbol read, stack symbol popped → stack symbol pushed)



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NPDA operation

- Taking a transition labeled:

  $a, b \rightarrow c$

  - $a \in (\Sigma \cup \{\varepsilon\})$
  - $b, c \in (\Gamma \cup \{\varepsilon\})$

  - read a from tape, or don't read from tape if $a = \varepsilon$
  - pop b from stack, or don't pop from stack if $b = \varepsilon$
  - push c onto stack, or don't push onto stack if $c = \varepsilon$

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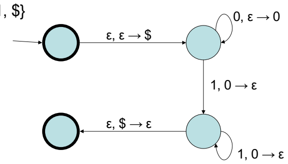
Example NPDA

Σ = {0, 1}

Γ = {0, 1, $}



- tape: 0 0 1 1   Stack contents: $

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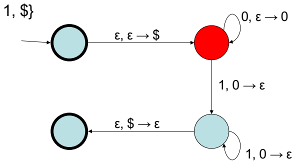
Example NPDA

Σ = {0, 1}

Γ = {0, 1, $}



- tape: 0 0 1 1   Stack contents: 0 $

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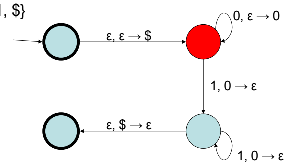
Example NPDA

Σ = {0, 1}

Γ = {0, 1, $}



- tape: 0 0 1 1   Stack contents: 0 0 $

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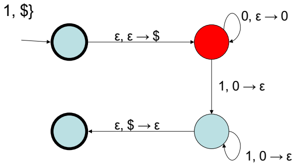
Example NPDA

Σ = {0, 1}

Γ = {0, 1, $}



- tape: 0 0 1 1   Stack contents: 0 0 $

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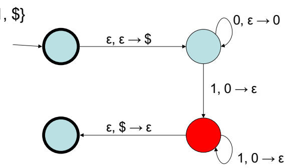
Example NPDA

Σ = {0, 1}

Γ = {0, 1, $}



- tape: 0 0 1 1   Stack contents: 0 $

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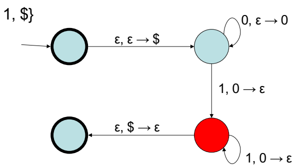
Example NPDA

Σ = {0, 1}

Γ = {0, 1, $}



- tape: 0 0 1 1   Stack contents: $
  accepted

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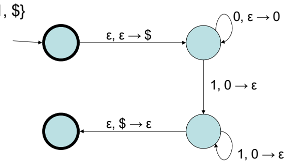
Example NPDA

$\Sigma = \{0, 1\}$

$\Gamma = \{0, 1, \$\}$



- tape: 0 0 1   Stack contents: $

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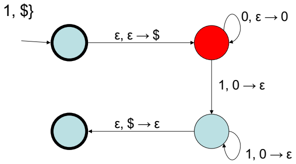
Example NPDA

$\Sigma = \{0, 1\}$

$\Gamma = \{0, 1, \$\}$



- tape: 0 0 1   Stack contents: 0 $

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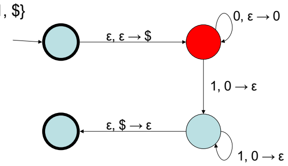
Example NPDA

$\Sigma = \{0, 1\}$

$\Gamma = \{0, 1, \$\}$



- tape: 0 0 1   Stack contents: 0 0 $

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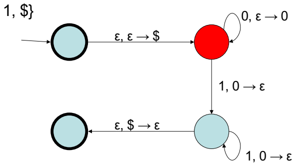
Example NPDA

$\Sigma = \{0, 1\}$

$\Gamma = \{0, 1, \$\}$



- tape: 0 0 1   Stack contents: 0 0 $

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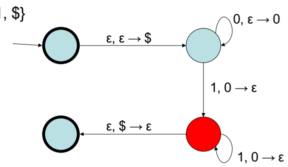
Example NPDA

$\Sigma = \{0, 1\}$

$\Gamma = \{0, 1, \$\}$



- tape: 0 0 1   Stack contents: 0 $
  not accepted

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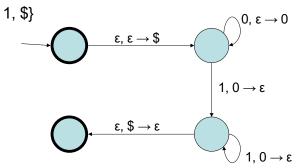
Example NPDA

$\Sigma = \{0, 1\}$

$\Gamma = \{0, 1, \$\}$



- What language does this NPDA accept?

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Formal definition of NPDA

- A NPDA is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where:
  - Q is a finite set called the states
  - Σ is a finite set called the tape alphabet
  - Γ is a finite set called the stack alphabet
  - $\delta: Q \times (\Sigma \cup \{\varepsilon\}) \times (\Gamma \cup \{\varepsilon\}) \to \mathcal{P}(Q \times (\Gamma \cup \{\varepsilon\}))$ is a function called the transition function
  - q_0 is an element of Q called the start state
  - F is a subset of Q called the accept states

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Formal definition of NPDA

- NPDA $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$ accepts string $w \in \Sigma^*$ if w can be written as $w_1w_2w_3\ldots w_m \in (\Sigma \cup \{\varepsilon\})^*$, and
- there exist states $r_0, r_1, r_2, \ldots, r_m$, and
- there exist strings $s_0, s_1, \ldots, s_m$ in $(\Gamma \cup \{\varepsilon\})^*$
  - $r_0 = q_0$ and $s_0 = \varepsilon$
  - $(r_{i+1}, b) \in \delta(r_i, w_{i+1}, a)$, where $s_i = at$, $s_{i+1} = bt$ for some $t \in \Gamma^*$
  - $r_m \in F$

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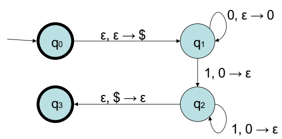
Example of formal definition



- $Q = \{q_0, q_1, q_2, q_3\}$
- $\Sigma = \{0,1\}$
- $\Gamma = \{0, 1, \$\}$
- $F = \{q_0, q_3\}$

- $\delta(q_0, \varepsilon, \varepsilon) = \{(q_1, \$)\}$
- $\delta(q_1, 0, \varepsilon) = \{(q_1, 0)\}$
- $\delta(q_1, 1, 0) = \{(q_2, \varepsilon)\}$
- $\delta(q_2, 1, 0) = \{(q_2, \varepsilon)\}$
- $\delta(q_2, \varepsilon, \$) = \{(q_3, \varepsilon)\}$

other values of $\delta(\bullet, \bullet, \bullet)$ equal {}

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Exercise

Design a NPDA for the language

$\{a^ib^jc^k : i, j, k \geq 0 \text{ and } i = j \text{ or } i = k\}$

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Context-free grammars and languages

- languages recognized by a (N)FA are exactly the languages described by regular expressions, and they are called the regular languages
- languages recognized by a NPDA are exactly the languages described by context-free grammars, and they are called the context-free languages

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