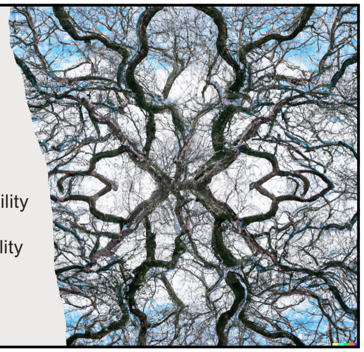
CS21
Decidability
and
Tractability

Lecture 3
January 8, 2024



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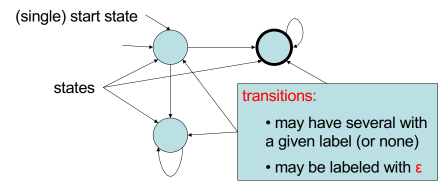
NFA diagrams



- At each step, several choices for next state
  - if *possible* to reach accept, then input accepted

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NFA formal definition

A nondeterministic FA ... (Q, Σ, δ ...

- Q is a finite set called ...
- Σ is a finite set called the alphabet
- δ:Q x (Σ ∪ {ε}) → 𝒫(Q) is a function called the transition function
- q_0 is an element of Q called the start state
- F is a subset of Q called the accept states

transit... labeled ... alpha... symbols or ε

“powerset of Q”: the set of all subsets of Q

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Formal description of NFA operation

NFA M = $(Q, \Sigma, \delta, q_0, F)$
accepts a string $w = w_1w_2w_3\ldots w_n \in \Sigma^*$
if w can be written (by inserting ε's) as:

$$y = y_1y_2y_3\ldots y_m \in (\Sigma \cup \{\varepsilon\})^*$$

and ∃ sequence $r_0, r_1, \ldots, r_m$ of states for which

- $r_0 = q_0$
- $r_{i+1} \in \delta(r_i, y_{i+1})$ for i = 0,1,2, …, m-1
- $r_m \in F$

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Closures

- Recall: want to show the set of languages recognized by NFA is closed under:
  - union “C = (A ∪ B)”
  - concatenation “C = (A ∘ B)”
  - star “ C = A* ”

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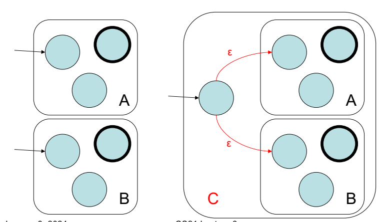
Closure under union

C = (A ∪ B) = {x : x ∈ A or x ∈ B}



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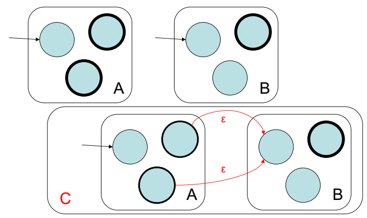
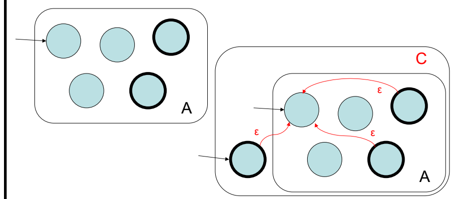
Closure under concatenation

$C = (A \circ B) = \{xy : x \in A \text{ and } y \in B\}$



Closure under star

$C = A^* = \{x_1x_2x_3\ldots x_k : k \geq 0 \text{ and each } x_i \in A\}$



NFA, FA equivalence

Theorem: a language L is recognized by a FA if and only if L is recognized by a NFA.

Must prove *two* directions:

(⇒) L is recognized by a FA implies L is recognized by a NFA.

(⇐) L is recognized by a NFA implies L is recognized by a FA.

(usually one is easy, the other more difficult)

NFA, FA equivalence

(⇒) L is recognized by a FA implies L is recognized by a NFA

Proof: a finite automaton *is* a nondeterministic finite automaton that happens to have no ε-transitions, and for which each state has exactly one outgoing transition for each symbol.

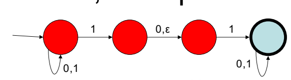
NFA, FA equivalence

(⇐) L is recognized by a NFA implies L is recognized by a FA.

Proof: we will build a FA that *simulates* the NFA (and thus recognizes the same language).

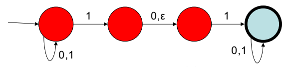
- alphabet will be the same
- what are the states of the FA?

NFA, FA equivalence



- given NFA $M = (Q, \Sigma, \delta, q_0, F)$
- construct FA $M' = (Q', \Sigma', \delta', q_0', F')$
- same alphabet: $\Sigma' = \Sigma$
- states are subsets of M's states: $Q' = \mathcal{P}(Q)$
- if we are in state $R \in Q'$ and we read symbol $a \in \Sigma'$, what is the new state?

NFA, FA equivalence



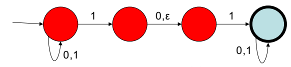
- given NFA M = (Q, Σ, δ, q_0, F)
- construct FA M' = (Q', Σ', δ', q_0', F')

Helpful def'n: E(S) = {q ∈ Q : q reachable from S by traveling along 0 or more ε-transitions}

- new transition fn: $\delta'(R, a) = \cup_{r \in R} E(\delta(r, a))$

= "all nodes reachable from R by following an a-transition, and then 0 or more ε-transitions"

NFA, FA equivalence



- given NFA M = (Q, Σ, δ, q_0, F)
- construct FA M' = (Q', Σ', δ', q_0', F')
- new start state: $q_0' = E(\{q_0\})$
- new accept states:
  F' = {R ∈ Q' : R contains an accept state of M)

NFA, FA equivalence

- We have proved (⇐) by construction.

  Formally we should also prove that the construction works, by induction on the number of steps of the computation.

  - at each step, the state of the FA M' is exactly the set of reachable states of the NFA M…

So far…

Theorem: the set of languages recognized by NFA is closed under union, concatenation, and star.

Theorem: a language L is recognized by a FA if and only if L is recognized by a NFA.

Theorem: the set of languages recognized by FA is closed under union, concatenation, and star.

Next…

- Describe the set of languages that can be built up from:
  - unions
  - concatenations
  - star operations
- Called "patterns" or regular expressions
- **Theorem**: a language L is recognized by a FA if and only if L is described by a regular expression.

Regular expressions

- R is a regular expression if R is
  - a, for some a ∈ Σ
  - ε, the empty string
  - Ø, the empty set
  - $(R_1 \cup R_2)$, where R_1 and R_2 are reg. exprs.
  - $(R_1 \circ R_2)$, where R_1 and R_2 are reg. exprs.
  - (R_1^*), where R_1 is a regular expression

A reg. expression R describes the language L(R).

Regular expressions

- example: $R = (0 \cup 1)$
  - if $\Sigma = \{0,1\}$ then use "Σ" as shorthand for R
- example: $R = 0 \circ \Sigma^*$
  - shorthand: omit "$\circ$" $R = 0\Sigma^*$
  - precedence: *, then $\circ$ then $\cup$, unless override by parentheses
  - in example $R = 0(\Sigma^*)$, not $R = (0\Sigma)^*$

Some examples

alphabet $\Sigma = \{0,1\}$

- {w : w has at least one 1}
  $= \Sigma^*1\Sigma^*$
- {w : w starts and ends with same symbol}
  $= 0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1$
- $\{w : |w| \leq 5\}$
  $= (\varepsilon \cup \Sigma)(\varepsilon \cup \Sigma)(\varepsilon \cup \Sigma)(\varepsilon \cup \Sigma)(\varepsilon \cup \Sigma)$
- {w : every 3^{rd} position of w is 1 starting with the first position}
  $= (1\Sigma\Sigma)^*(\varepsilon \cup 1 \cup 1\Sigma)$

Manipulating regular expressions

- The empty set and the empty string:
  - $R \cup \emptyset = R$
  - $R\varepsilon = \varepsilon R = R$
  - $R\emptyset = \emptyset R = \emptyset$
  - $\cup$ and $\circ$ behave like +, x; $\emptyset$, ε behave like 0,1
- additional identities:
  - $R \cup R = R$ (here + and $\cup$ differ)
  - $(R_1^*R_2)^*R_1^* = (R_1 \cup R_2)^*$
  - $R_1(R_2R_1)^* = (R_1R_2)^*R_1$