CS21 Decidability and Tractability

Lecture 22
February 26, 2024

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Outline

- NP-complete problems: independent set, vertex cover, clique…
- NP-complete problems: Hamilton path and cycle,Traveling Salesperson Problem
- NP-complete problems: Subset Sum
- NP-complete problems: NAE-3-SAT, max cut

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SUBSET-SUM is NP-complete

Theorem: the following language is NP-complete:

SUBSET-SUM = { … B) :
there is a sub…

our reduction had better produce super-polynomially large B (unless we want to prove P=NP)

- Proof:
  - Part 1: SUBSET-S… ∈ NP. Proof:
  - Part 2: SUBSET-SUM is NP-hard.
    - reduce from?

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SUBSET-SUM is NP-complete

- We are reducing from the language:

3SAT = { φ : φ is a 3-CNF formula that has a satisfying assignment }

to the language:

SUBSET-SUM = {(S = {a_1, a_2, a_3, …, a_k}, B) : there is a subset of S that sums to B}

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SUBSET-SUM is NP-complete

- $\varphi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_4 \vee x_3) \wedge \ldots \wedge (\ldots)$
- Need integers to play the role of truth assignments
- For each variable x_i include two integers in our set S:
  - x_i^{TRUE} and x_i^{FALSE}
- set B so that exactly one must be in sum

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SUBSET-SUM is NP-complete

x_1^{TRUE} = 1 0 0 0 … 0
x_1^{FALSE} = 1 0 0 0 … 0
x_2^{TRUE} = 0 1 0 0 … 0
x_2^{FALSE} = 0 1 0 0 … 0
…
x_m^{TRUE} = 0 0 0 0 … 1
x_m^{FALSE} = 0 0 0 0 … 1
B = 1 1 1 1 … 1

- every choice of one from each (x_i^{TRUE},x_i^{FALSE}) pair sums to B
- every subset that sums to B must choose one from each (x_i^{TRUE},x_i^{FALSE}) pair

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SUBSET-SUM is NP-complete

- $\varphi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_4 \vee x_3) \wedge \ldots \wedge (\ldots)$
- Need to force subset to "choose" at least one true literal from each clause
- Idea:
  - add more digits
  - one digit for each clause
  - set B to force each clause to be satisfied.

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SUBSET-SUM is NP-complete

- $\varphi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_4 \vee x_3) \wedge \ldots \wedge (\ldots)$

x_1^{TRUE} = 1 0 0 0 … 0 1
x_1^{FALSE} = 1 0 0 0 … 0 0
x_2^{TRUE} = 0 1 0 0 … 0 1
x_2^{FALSE} = 0 1 0 0 … 0 0
x_3^{TRUE} = 0 0 1 0 … 0 0
x_3^{FALSE} = 0 0 1 0 … 0 1
…
B = 1 1 1 1 … 1 ? ? ? ?

clause 1
clause 2
clause 3
:
clause k

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SUBSET-SUM is NP-complete

- B = 1 1 1 1 … 1 ? ? ? ?
- if clause i is satisfied sum might be 1, 2, or 3 in corresponding column.
- want ? to "mean" ≥ 1
- solution: set ? = 3
- add two "filler" elements for each clause i:
- $FILL1_i$ = 0 0 0 0 … 0 0 … 0 1 0 … 0
- $FILL2_i$ = 0 0 0 0 … 0 0 … 0 1 0 … 0

  column for clause i

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SUBSET-SUM is NP-complete

- Reduction: m variables, k clauses
  - for each variable x_i:
    - x_i^{TRUE} has ones in positions k + i and {j : clause j includes literal x_i}
    - x_i^{FALSE} has ones in positions k + i and {j : clause j includes literal $\neg x_i$}
  - for each clause i:
    - $FILL1_i$ and $FILL2_i$ have one in position i
  - bound B has 3 in positions 1…k and 1 in positions k+1…k+m

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SUBSET-SUM is NP-complete

- Reduction computable in poly-time?
- YES maps to YES?
  - choose one from each (x_i^{TRUE},x_i^{FALSE}) pair corresponding to a satisfying assignment
  - choose 0, 1, or 2 of filler elements for each clause i depending on whether it has 3, 2, or 1 true literals
  - first m digits add to 1; last k digits add to 3

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SUBSET-SUM is NP-complete

- NO maps to NO?
  - at most 5 ones in each column, so no carries to worry about
  - first m digits of B force subset to choose exactly one from each (x_i^{TRUE},x_i^{FALSE}) pair
  - last k digits of B require at least one true literal per clause, since can only sum to 2 using filler elements
  - resulting assignment must satisfy φ

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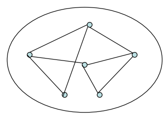
MAX CUT

- Given graph $G = (V, E)$
  - a cut is a subset $S \subseteq V$
  - an edge (x, y) crosses the cut if $x \in S$ and $y \in V - S$ or $x \in V - S$ and $y \in S$
  - search problem:

    find cut maximizing number of edges crossing the cut



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MAX CUT

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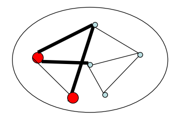

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MAX CUT

Theorem: the following language is NP-complete:

MAX CUT = {(G = (V, E), k) : there is a cut $S \subseteq V$ with at least k edges crossing it}

- Proof:
  - Part 1: MAX CUT $\in$ NP. Proof?
  - Part 2: MAX CUT is NP-hard.
    - reduce from?

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Not-All-Equal 3SAT

$(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_4 \vee x_3) \wedge \ldots \wedge (\ldots)$

Theorem: the following language is NP-complete:

NAE3SAT = {φ : φ is a 3-CNF formula for which there exists a truth assignment in which every clause has at least 1 true literal and at least 1 false literal}

- Proof:
  - Part 1: NAE3SAT $\in$ NP. Proof?
  - Part 2: NAE3SAT is NP-hard. Reduce from?

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NAE3SAT is NP-complete

- We are reducing from the language:

  CIRCUIT-SAT = {C : C is a Boolean circuit for which there exists a satisfying truth assignment}

  to the language:

  NAE3SAT = {φ : φ is a 3-CNF formula for which there exists a truth assignment in which every clause has at least 1 true literal and at least 1 false literal}

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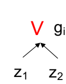
NAE3SAT is NP-complete

- Recall reduction to 3SAT
  - variables $x_1, x_2, \ldots, x_n$, gates $g_1, g_2, \ldots, g_m$
  - produce clauses:

not all true in a satisfying assignment



- $(\neg z_1 \vee g_i)$
- $(\neg z_2 \vee g_i)$
- $(\neg g_i \vee z_1 \vee z_2)$

- $(\neg g_i \vee z_1)$
- $(\neg g_i \vee z_2)$
- $(\neg z_1 \vee \neg z_2 \vee g_i)$

- $(g_i \vee z)$
- $(\neg z \vee \neg g_i)$

output gate g_m:

- (g_m)

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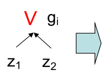
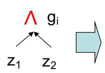
NAE3SAT is NP-complete

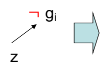
- Recall reduction to 3SAT
  - variables $x_1, x_2, \ldots, x_n$, gates $g_1, g_2, \ldots, g_m$
  - produce clauses:



- $(\neg z_1 \vee g_i \vee w)$
- $(\neg z_2 \vee g_i \vee w)$
- $(\neg g_i \vee z_1 \vee z_2)$



- $(\neg g_i \vee z_1 \vee w)$
- $(\neg g_i \vee z_2 \vee w)$
- $(\neg z_1 \vee \neg z_2 \vee g_i)$



- $(g_i \vee z \vee w)$
- $(\neg z \vee \neg g_i \vee w)$

output gate g_m:

- $(g_m \vee w)$

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NAE3SAT is NP-complete

- Does the reduction run in polynomial time?
- YES maps to YES
  - already know how to get a satisfying assignment to the BLUE variables
  - set w = FALSE

- $(\neg z_1 \vee g_i \vee w)$
- $(\neg z_2 \vee g_i \vee w)$
- $(\neg g_i \vee z_1 \vee z_2)$
- $(\neg g_i \vee z_1 \vee w)$
- $(\neg g_i \vee z_2 \vee w)$
- $(\neg z_1 \vee \neg z_2 \vee g_i)$
- $(g_i \vee z \vee w)$
- $(\neg z \vee \neg g_i \vee w)$
- $(g_m \vee w)$

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NAE3SAT is NP-complete

- NO maps to NO
  - given NAE assignment A
  - complement A' is a NAE assignment
  - A or A' has w = FALSE
  - must have TRUE BLUE variable in every clause
  - we know this implies C satisfiable

- $(\neg z_1 \vee g_i \vee w)$
- $(\neg z_2 \vee g_i \vee w)$
- $(\neg g_i \vee z_1 \vee z_2)$
- $(\neg g_i \vee z_1 \vee w)$
- $(\neg g_i \vee z_2 \vee w)$
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- $(g_m \vee w)$

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MAX CUT

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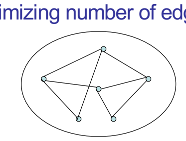

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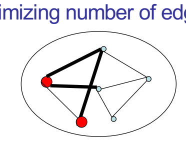

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- Proof:
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MAX CUT is NP-complete

- We are reducing from the language:

NAE3SAT = {φ : φ is a 3-CNF formula for which there exists a truth assignment in which every clause has at least 1 true literal and at least 1 false literal}

to the language:

MAX CUT = {(G = (V, E), k) : there is a cut S ⊆ V with at least k edges crossing it}

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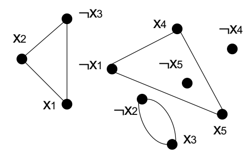
MAX CUT is NP-complete

- The reduction:
  – given instance of NAE3SAT (n nodes, m clauses):

$(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_4 \vee x_5) \wedge \ldots \wedge (\neg x_2 \vee x_3 \vee x_3)$

  – produce graph G = (V, E) with node for each literal



- triangle for each 3-clause
- parallel edges for each 2-clause

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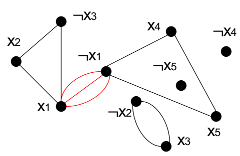
MAX CUT is NP-complete



- triangle for each 3-clause
- parallel edges for each 2-clause

  – if cut selects TRUE literals, each clause contributes 2 if NAE, and < 2 otherwise
  – need to penalize cuts that correspond to inconsistent truth assignments
  – add n_i parallel edges from x_i to $\neg x_i$ (n_i = # occurrences)
  (repeat variable in 2-clause to make 3-clause for this calculation)

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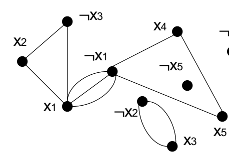
MAX CUT is NP-complete



- triangle for each 3-clause
- parallel edges for each 2-clause
- n_i parallel edges from x_i to $\neg x_i$
- set k = 5m

- YES maps to YES
  – take cut to be TRUE literals in a NAE truth assignment
  – contribution from clause gadgets: 2m
  – contribution from $(x_i, \neg x_i)$ parallel edges: 3m

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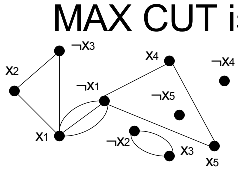
MAX CUT is NP-complete



- triangle for each 3-clause
- parallel edges for each 2-clause
- n_i parallel edges from x_i to $\neg x_i$
- set k = 5m

- NO maps to NO
  – **Claim**: if cut has x_i, $\neg x_i$ on same side, then can move one to opposite side without decreasing # edges crossing cut
  – contribution from $(x_i, \neg x_i)$ parallel edges: 3m
  – contribution from clause gadgets must be 2m
  – conclude: there is a NAE assignment

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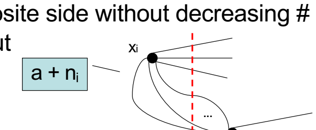
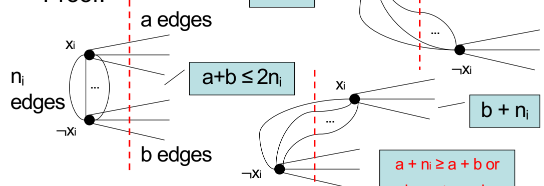
MAX CUT is NP-complete

Claim: if cut has x_i, $\neg x_i$ on same side, then can move one to opposite side without decreasing # edges crossing cut

- Proof:





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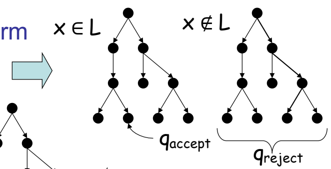
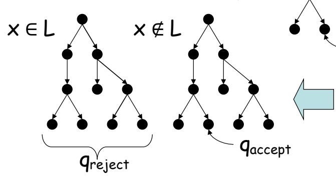
coNP

- Is NP closed under complement?

Can we transform this machine:





into a machine with this behavior?

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coNP

- language L is in coNP iff its complement (co-L) is in NP
- it is believed that NP ≠ coNP
- note: P = NP implies NP = coNP
  - proving NP ≠ coNP would prove P ≠ NP
  - another major open problem…

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coNP

- canonical coNP-complete language:

UNSAT = {φ : φ is an unsatisfiable 3-CNF formula}

  - proof?

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coNP

Disjunctive Normal Form = OR of ANDs

- another example

3-DNF-TAUTOLOGY = {φ : φ is a 3-DNF formula and for all x, φ(x) =1}

  - proof?
- another example:

EQUIV-CIRCUIT = {(C_1, C_2) : C_1 and C_2 are Boolean circuits and for all x, C_1(x) = C_2(x)}

  - proof?

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