CS21
Decidability and Tractability

Lecture 11
January 29, 2024

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## Examples of basic operations

- Convince yourself that the following types of operations are easy to implement as part of TM "program"
  (but perhaps tedious to write out…)
  - copying
  - moving
  - incrementing/decrementing
  - arithmetic operations +, -, *, /



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## Universal TMs and encoding

- the input to a TM is always a string in $\Sigma^*$
- often we want to interpret the input as representing another object
- examples:
  - tuple of strings (x, y, z)
  - 0/1 matrix
  - graph in adjacency-list format
  - Context-Free Grammar



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## Universal TMs and encoding

- the input to a TM is always a string in $\Sigma^*$
- we must encode our input as such a string
- examples:
  - tuples separated by #: #x#y#z
  - 0/1 matrix given by: #n#x# where $x \in \{0,1\}^{n^2}$
- any reasonable encoding is OK
- emphasize "encoding of X" by writing <X>



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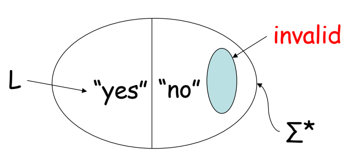
## Universal TMs and encoding

- some strings not valid encodings and these are not in the language




make sure TM can recognize invalid encodings and reject them



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## Universal TMs and encoding

- We can easily construct a Universal TM that recognizes the language:
  $A_{TM}$ = {<M, w> : M is a TM and M accepts w}
  - how?
- this is a remarkable feature of TMs (not possessed by FA or NPDAs…)
- means there is a general purpose TM whose input can be a "program" to run



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# Church-Turing Thesis

- many other models of computation
  - we saw multitape TM, nondeterministic TM
  - others don't resemble TM at all
  - common features:
    - unrestricted access to unlimited memory
    - finite amount of work in a single step
- every single one can be simulated by TM
- many are equivalent to a TM
- problems that can be solved by computer does not depend on details of model!



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# Church-Turing Thesis

- the belief that TMs formalize our intuitive notion of an algorithm is:

The Church-Turing Thesis

everything we can compute on a physical computer

can be computed on a Turing Machine

- Note: this is a belief, not a theorem.



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# Recursive Enumerability

- Why is "Turing-recognizable" called RE?
- Definition: a language $L \subseteq \Sigma^*$ is recursively enumerable if there is exists a TM (an "enumerator") that writes on its output tape

$$\#x_1\#x_2\#x_3\#\ldots$$

and $L = \{x_1, x_2, x_3, \ldots\}$.

- The output may be infinite



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# Recursive Enumerability

**Theorem**: A language is Turing-recognizable iff some enumerator enumerates it.

Proof:

($\Leftarrow$) Let E be the enumerator. On input w:

- Simulate E. Compare each string it outputs with w.
- If w matches a string output by E, accept.



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# Recursive Enumerability

**Theorem**: A language is Turing-recognizable iff some enumerator enumerates it.

Proof:

($\Rightarrow$) Let M recognize language $L \subseteq \Sigma^*$.

- let $s_1, s_2, s_3, \ldots$ be enumeration of $\Sigma^*$ in lexicographic order.
- for i = 1,2,3,4,…
  - simulate M for i steps on $s_1, s_2, s_3, \ldots, s_i$
- if any simulation accepts, print out that $s_j$



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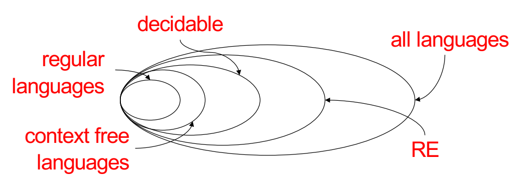
# Undecidability




decidable $\subseteq$ RE $\subseteq$ all languages

our goal: prove these containments proper



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## Countable and Uncountable Sets

- the natural numbers **N** = {1,2,3,…} are countable
- Definition: a set S is countable if it is finite, or it is infinite and there is a bijection

  f: **N** → S



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## Countable and Uncountable Sets

- Theorem: the positive rational numbers

  Q = {m/n : m, n ∈ **N** } are countable.
- Proof:

…

1/1 1/2 1/3 1/4 1/5 1/6 …
2/1 2/2 2/3 2/4 2/5 2/6 …
3/1 3/2 3/3 3/4 3/5 3/6 …
4/1 4/2 4/3 4/4 4/5 4/6 …
5/1 …



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## Countable and Uncountable Sets

**Theorem**: the real numbers **R** are NOT countable (they are "uncountable").

- How do you prove such a statement?
  - assume countable (so there exists bijection f)
  - derive contradiction (some element not mapped to by f)
  - technique is called diagonalization (Cantor)



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## Countable and Uncountable Sets

- Proof:
  - suppose **R** is countable
  - list **R** according to the bijection f:

| n | f(n) |
|---|---|
| 1 | 3.14159… |
| 2 | 5.55555… |
| 3 | 0.12345… |
| 4 | 0.50000… |
| … | |



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## Countable and Uncountable Sets

- Proof:
  - suppose **R** is countable
  - list **R** according to the bijection f:

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| … | |

set $x = 0.a_1a_2a_3a_4\ldots$

where digit $a_i \neq i^{th}$ digit after decimal point of f(i) (not 0, 9)

e.g. x = 0.2312…

x cannot be in the list!



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## non-RE languages

**Theorem**: there exist languages that are not Recursively Enumerable.

Proof outline:
- the set of all TMs is countable
- the set of all languages is uncountable
- the function L:{TMs} →{languages} cannot be onto



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# non-RE languages

- Lemma: the set of all TMs is countable.
- Proof:
  – each TM M can be described by a finite-length string <M>
  – can enumerate these strings, and give the natural bijection with **N**



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# non-RE languages

- Lemma: the set of all languages is uncountable
- Proof:
  – fix an enumeration of all strings $s_1, s_2, s_3, \ldots$
  (for example, lexicographic order)
  – a language L is described by its characteristic vector $\chi_L$ whose $i^{th}$ element is 0 if $s_i$ is not in L and 1 if $s_i$ is in L



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# non-RE languages

– suppose the set of all languages is countable
– list characteristic vectors of all languages according to the bijection f:

| n | f(n) |
|---|---|
| 1 | 0101010… |
| 2 | 1010011… |
| 3 | 1110001… |
| 4 | 0100011… |
| … | |



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# non-RE languages

– suppose the set of all languages is countable
– list characteristic vectors of all languages according to the bijection f:

| n | f(n) |
|---|---|
| 1 | 0101010… |
| 2 | 1010011… |
| 3 | 1110001… |
| 4 | 0100011… |
| … | |

set x = 1101…

where $i^{th}$ digit ≠ $i^{th}$ digit of f(i)

x cannot be in the list!

therefore, the language with characteristic vector x is not in the list



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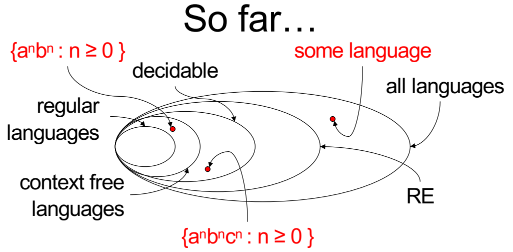
# So far…




- This language might be an esoteric, artificially constructed one. Do we care?
- We will show a natural undecidable L next.



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# The Halting Problem

- Definition of the "Halting Problem":
  HALT = { <M, x> : TM M halts on input x }
- HALT is recursively enumerable.
  – proof?
- Is HALT decidable?



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