CS 153 Current Topics in Theoretical Computer Science **Spring 2016**

Problem Set 1

Out: May 10 *Due:* **May 24**

You are encouraged to work in groups of two or three; however you must turn in your own write-up and note with whom you worked. You may consult the course notes and resources linked from the webpage. Any other general informational resources (such as Wikipedia pages) is fine but please do not seek out solutions to the specific problems.

1. Suppose that for all sufficiently large integers $n > 0$, there exist sets $S \subseteq Z_n^N$ of cardinality $n^{(1-o(1))\cdot N}$ with the property that: for all $s, t, u \in S$, not all equal, there exists an i for which $|\{s_i, t_i, u_i\}| = 2$. Show that $\omega = 2$. Hint: find a way, using S, to make use of the optimal capacity USPs we constructed in class.

2. We can define a non-abelian version of the 2-families construction. These are subsets

   $$A_1, A_2, \dots A_n, B_1, B_2, \dots, B_n \subseteq G$$

   satisfying (1) for all i, $|A_iB_i| = |A_i||B_i|$ and (2) for all i and $j \neq k$, $A_iB_i \cap A_jB_k = \emptyset$.

   (a) Give an optimal construction in the symmetric group: that is, in $G = S_k$ give a construction with $n = |S_k|^{1/2-o(1)}$ and $|A_i| = |B_i| = |S_k|^{1/2-o(1)}$.

   (b) Let G be the semidirect product of Z_{10n}^n with S_n in which S_n acts by permuting the coordinates. Give an optimal construction in G. Hint: the sets will be indexed by elements of a 3-term-arithmetic-progression-free set $S \subseteq Z_{10n}^n$.

3. Prove a lower bound of 6 on the s-rank of $\langle 2, 2, 2\rangle$.

4. It is a prominent open problem in combinatorics to prove or disprove the existence of subsets $S \subseteq Z_3^n$ with $|S| \geq 3^{(1-o(1))\cdot n}$, satisfying $s + t + u = 0 \Leftrightarrow s = t = u$ for all $s, t, u \in S$. If one is interested in constructing such sets, here is a (potentially) easier problem: find a set T of *triples* of elements from Z_3^n satisfying (1) for all $(t_1, t_2, t_3) \in T$, $t_1 + t_2 + t_3 = 0$, and (2) for all $s = (s_1, s_2, s_3), t = (t_1, t_2, t_3), u = (u_1, u_2, u_3) \in T$

   $$s_1 + t_2 + u_3 = 0 \Leftrightarrow s = t = u.$$

   (a) Show that a set S satisfying $s + t + u = 0 \Leftrightarrow s = t = u$ for all $s, t, u \in S$ implies a set T with the same size satisfying the above two properties.

   (b) Show that if there exists a strong USP U with w columns and N rows, then there exist sets T satisfying the above two properties with cardinality $N2^{2w/3}$. You may assume U is *balanced*, meaning that each row has equal numbers of 1's, 2's and 3's. You will need the following lemma about large *diagonals* in matrix multiplication tensors:

Lemma 1.1 *There exists a subset D of the support of $\langle n, n, n\rangle$ (which is a subset of $[n]^3$) with cardinality at least $\Omega(n^2)$, and with the property that for $(i, j, k) \in D$, knowing any two of $\{i, j, k\}$ determines the third.*

(c) Using this connection, for what capacity C can you construct sets T of cardinality C^n satisfying the above two properties unconditionally? What about if there exist strong USPs with (the optimal) $N = \binom{w}{w/3}^{1-o(1)}$ rows?