CS 153 Current topics in theoretical computer science	**Spring 2012**

Solution Set 1

Out: May 15

1. We use the substitution method. By setting $b_{1,1} = 1$ and $b_{2,1} = 0$, we see that some product of a's and b's depends on $a_{1,1}$; set $a_{1,1}$ to a linear form in the a's that makes this product zero. Then by setting $b_{1,1} = 0$ and $b_{2,1} = 1$, we see that some product of a's and b's depends on $a_{1,2}$; set $a_{1,2}$ to a linear form in the a's that makes this product zero. What remains still computes a 1×2 by 2×2 matrix multiplication. Since $\langle 1, 2, 2\rangle$ has 4 linearly independent slices, there must be at least 4 remaining slices.

2. Set $\alpha_i = \log_{|G_i|}(|X_i| \cdot |Y_i| \cdot |Z_i|)$. From a theorem proved in class, we have that for each i,
$$(|X_i||Y_i||Z_i|)^{\omega/3} \leq D_i^{\omega-2}|G_i|.$$
Taking logs and dividing by $\log|G_i|$ we get
$$\alpha_i\omega/3 \leq (1/2 - \epsilon)(\omega - 2) + 1$$
Replacing α_i with its supremum, we get
$$\omega/2 \leq (1/2 - \epsilon)(\omega - 2) + 1$$
which simplifies to $\epsilon\omega \leq 2\epsilon$, hence $\omega = 2$.

3. Recall that we found three subsets X, Y, Z of S_n with $|X| = |Y| = |Z| = |S_n|^{1/2-o(1)}$. For each $y \in Y$, define:
$$\begin{aligned} A_y &= \{xy^{-1} : x \in X\} \\ B_y &= \{yz^{-1} : z \in Z\} \end{aligned}$$
Observe that $A_yB_y = \{xz^{-1} : x \in X, z \in Z\}$, and these must all be distinct; if we had $(x, z) \neq (x', z')$ with $xz^{-1} = x'(z')^{-1}$ then $(x')^{-1}xz^{-1}z' = 1$ which violates the triple product property (since $1 \in Q(Y)$). Also, if $A_uB_u \cap A_yB_{y'}$ with $y \neq y'$, then we have $xz^{-1} = x'y^{-1}y'(z')^{-1}$, and thus $x^{-1}x'y^{-1}y'(z')^{-1}z = 1$, which violates the triple product property.

4. Such a table has three types of columns – columns containing only 1's and 2's, columns containing only 2's and 3's, and columns containing only 3's and 1's. Let n_1, n_2, n_3 denote the number of each type of column. The number of distinct 1/2 patterns is 2^{n_1}, the number of distinct 2/3 patterns is 2^{n_2} and the number of distinct 3/1 patterns is 2^{n_3}. If $N > 2^{n_1}2^{n_2}$ then by the pigeonhole principle, there are two rows with identical 1/2 patterns in the 1/2 columns and 2/3 patterns in the 2/3 columns. Thus these two rows have identical "2-sets", and thus the table is not a strong USP. Thus $N \leq 2^{n_1+n_2}$. Similarly, the fact that a USP can have no duplicate "1-sets" implies $N \leq 2^{n_1+n_3}$, and the fact that a USP can have no duplicate "2-sets" implies that $N \leq 2^{n_2+n_3}$. Thus $N^3 \leq 2^{2(n_1+n_2+n_3)} = 2^{2n}$.

5. (a) Consider the matrix M with $M[i,j] = \omega^{i+j}$, where ω is a primitive n-th root of unity, and note that M has rank 1. Let J be the all-ones matrix (also rank 1). Then $M - I$ has rank at most 2, and it has the same support as $J - I$, which has rank $n - 1$.

   (b) We first show that $R(T) = 3$. Clearly it is at most 3. Now suppose for the purpose of contradiction that a_1 and a_2 were spanned by two rank one slices b_1 and b_2. It cannot be the case that both b_1 and b_2 have 0 in the upper left corner since then they would not span a_1. It cannot be the case that exactly one has a 0 in the upper left corner, because then that slice must equal a_2 (which is not a rank one slice). So (after scaling) we must have

$$b_1 = \begin{array}{|c|c|}\hline 1 & x \\ \hline y & xy \\ \hline \end{array} \quad b_2 = \begin{array}{|c|c|}\hline 1 & s \\ \hline t & st \\ \hline \end{array}$$

from which we get the equation:

$$\begin{pmatrix} 1 & 1 \\ x & s \\ y & t \\ xy & st \end{pmatrix} \cdot M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix},$$

where M is a 2×2 matrix. Thus M must be full rank, and so $(y,t) = (xy, st)$. This equation leaves us with four possibilities: (1) $y = 0$ and $t = 0$; (2) $y = 0$ and $s = 1$; (3) $x = 1$ and $t = 0$; (4) $x = 1$ and $s = 1$. In case (1) we have $(0,0)M = (0,1)$ from the third row of the linear system, a contradiction. In case (4) we have we have $(1,1)M = (1,0)$ from the first row of the linear system and $(1,1)M = (1,1)$ from the second row of the linear system, a contradiction. In case (2), we know that $M = (1,1;0,t)^{-1} = (1,-1/t;0,1/t)$ from the first and third rows. But then $(x,1)M = (1,1)$ implies $x = 1$, and this is a contradiction, since M has full rank (from the upper half of the linear system). Case (3) is symmetric; i.e., we know that $M = (1,1;y,0)^{-1} = (0,1/y;1,1/y)$ from the first and third rows. But then $(1,s)M = (1,1)$ implies $s = 1$, and this is a contradiction. We conclude that $R(T) > 2$.

Finally, the pseudorank of T is 2. Consider the slices

$$b_1 = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0 & 0 \\ \hline \end{array} \quad b_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & 2 \\ \hline \end{array}$$

Note that a_1 has the same support as b_1 and a_2 has the same support as $b_2 - b_1$.