Variational integrators for electric circuits

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Variational integrators are symplectic-momentum preserving integrators that are based on a discrete variational formulation of the underlying system. So far, variational integrators have been mainly developed and used for a wide variety of mechanical systems. In this work, we develop a variational integrator for the simulation of electric circuits. An appropriate variational formulation is presented to model the circuit from which the equations of motion are derived. Finally, a corresponding time-discrete variational formulation provides an iteration scheme for the simulation of the electric circuit. In this way, a variational integrator is constructed that gains several advantages. A comparison to standard integration techniques shows that even for simple LCR circuits a better long-time energy behavior and frequency preservation can be obtained.

1 Variational formulation for electric circuits

Consider a circuit as a connected, directed graph with n edges and $m + 1$ nodes (and $n - m$ meshes for planar graphs) (cf. Figure 1 (left)). On the ith edge, there are: a capacitor with capacitance C_i, an inductor with inductance L_i, a voltage source ϵ_i and a resistor with resistance R_i, one or several of which can be zeros. Thus, branches in the circuit correspond to edges in the graph. Let $q(t), v(t), u(t), p(t) \in \mathbb{R}^n$ be the time-dependent charges, currents, voltages, and flux linkages of the circuit elements. Furthermore, let $K \in \mathbb{R}^{n,m}$ and $K_2 \in \mathbb{R}^{n,n-m}$ be the *Kirchhoff Constraint matrix* and the *Fundamental Loop matrix* of a given circuit. The Kirchhoff current and voltage constraints are given as $K^T v = 0$ (or $K_2 \tilde{v} = v$ with the mesh currents $\tilde{v}$) and $K_2^T u = 0$ (or $K\hat{u} = u$ with the node voltages $\hat{u}$). Note that $ker(K_2^T) = im(K)$. For simplicity we focus on ideal linear circuit elements with linear constitutive laws.

We can define a Lagrangian $\mathcal{L}(q, v) = \frac{1}{2} v^T L v - \frac{1}{2} q^T C q$ of the circuit system consisting of the difference between magnetic and electric energy with $L = \text{diag}(L_1, \ldots, L_n)$ and $C = \text{diag}\left(\frac{1}{C_1}, \ldots, \frac{1}{C_n}\right)$. In the case where no inductor (resp. no capacitor) is on branch i, it holds $L_i = 0$ (resp. $\frac{1}{C_i} = 0$). Note that the Lagrangian can be degenerate if the Legendre transform is not invertible, i.e. L is singular. The Lagrangian force of the system $f_L(q, v, u) = -\text{diag}(R)v + \text{diag}(\mathcal{E})u$ involves resistors and external voltage sources with $R = (R_1, \ldots, R_n)^T$ and $\mathcal{E} = (\epsilon_1, \ldots, \epsilon_n)^T$. If no resistor (resp. no voltage source) is on branch i, it holds $R_i = 0$ (resp. $\epsilon_i = 0$). In presence of a voltage source on branch i, it holds $\epsilon_i = 1$.

To reduce the system's dimension, we define the Lagrangian and the forces on the space of meshes. The branch currents and charges are given by the mesh currents and charges as $v = K_2 \tilde{v}$ and $q = K_2 \tilde{q}$ for consistent initial values. We define the reduced Lagrangian in terms of the mesh variables as $\mathcal{L}^M(\tilde{q}, \tilde{v}) = \mathcal{L}(K_2\tilde{q}, K_2\tilde{v}) = \frac{1}{2}\tilde{v}^T K_2^T L K_2 \tilde{v} - \frac{1}{2}\tilde{q}^T K_2^T C K_2 \tilde{q}$. The reduced force is defined as $f_L^M(\tilde{q}, \tilde{v}, u) = K_2^T f_L(K_2\tilde{q}, K_2\tilde{v}, u) = -K_2^T \text{diag}(R)K_2\tilde{v} + K_2^T \text{diag}(\mathcal{E})u$. Defining the mesh fluxes as $\tilde{p} = K_2^T p \in \mathbb{R}^{n-m}$ we can derive the equations of motion for the circuit system with the Lagrange-d'Alembert-Pontryagin principle [2], i.e. we are searching for curves $\tilde{q}(t), \tilde{v}(t)$ and $\tilde{p}(t)$ fulfilling

$$\delta \int_0^T \mathcal{L}^M(\tilde{q}(t), \tilde{v}(t)) + \left\langle \tilde{p}(t), \dot{\tilde{q}}(t) - \tilde{v}(t) \right\rangle \, dt + \int_0^T f_L^M(\tilde{q}(t), \tilde{v}(t), u(t)) \cdot \delta\tilde{q}(t) \, dt = 0 \tag{1}$$

with fixed initial and final variations $\delta\tilde{q}(0) = \delta\tilde{q}(T) = 0$. Taking arbitrary variations $\delta\tilde{v}$ and $\delta\tilde{p}$ and $\delta\tilde{q}$ we obtain the Euler-Lagrange equations $\frac{\partial \mathcal{L}^M}{\partial \tilde{q}} - \dot{\tilde{p}} + f_L^M = 0$, $\dot{\tilde{q}} = \tilde{v}$, $\frac{\partial \mathcal{L}^M}{\partial \tilde{v}} - \tilde{p} = 0$, which simplify for the linear case to the DAE-system

$$\dot{\tilde{p}} = K_2^T \left(-C K_2 \tilde{q} - \text{diag}(R) K_2 \tilde{v} + K_2^T \text{diag}(\mathcal{E})u \right), \qquad \dot{\tilde{q}} = \tilde{v}, \qquad \tilde{p} = K_2^T L K_2 \tilde{v}. \tag{2}$$

If the matrix $K_2^T L K_2$ is regular, the variable $\tilde{v}$ can be eliminated in the algebraic equation. In this case, the Euler-Lagrange equations (2) represent a non-degenerate Lagrangian system. The following proposition holds (in an analogous way a similar statement also holds for RCL and RCLV circuits). For a proof we refer to [3].

Proposition 1.1 *For LC circuits (including only self inductors), the system is non-degenerate if the number of capacitors equals the number of independent constraints involving the currents through the capacitives' branches.*

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2 Variational integrator and numerical results

For the discrete variational derivation (cf. [1]), we introduce a discrete time grid $\Delta t = \{t_k = kh \,|\, k = 0, \ldots, N\}$, $Nh = T$, $N \in \mathbb{N}$. We replace the charge by its discrete version $\tilde{q}_d(\Delta t)$, where we view $\tilde{q}_k = \tilde{q}_d(kh)$ as an approximation to $\tilde{q}(kh)$ (analogously for $\tilde{v}(t), \tilde{p}(t), u(t)$). We replace the reduced Lagrange-d'Alembert-Pontryagin principle with a discrete version

$$\delta \left\{ h \sum_{k=0}^{N-1} \left(\mathcal{L}^M(\tilde{q}_k, \tilde{v}_k) + \left\langle \tilde{p}_k, \frac{\tilde{q}_{k+1} - \tilde{q}_k}{h} - \tilde{v}_k \right\rangle \right) \right\} + h \sum_{k=0}^{N-1} f_L^M(\tilde{q}_k, \tilde{v}_k, u_k) \delta \tilde{q}_k = 0, \tag{3}$$

where the time derivative $\dot{q}(t)$ is approximated by the forward difference operator. Taking discrete variations $\delta \tilde{q}_k$, $\delta \tilde{v}_k$, and $\delta \tilde{p}_k$ with $\delta \tilde{q}_0 = \delta \tilde{q}_N = 0$ leads to the discrete Euler-Lagrange equations, which read for the linear case as

$$\frac{\tilde{p}_k - \tilde{p}_{k-1}}{h} = K_2^T \left(-C K_2 \tilde{q}_k - \mathrm{diag}(R) K_2 \tilde{v}_k + K_2^T \mathrm{diag}(\mathcal{E}) u_k \right), \qquad \frac{\tilde{q}_k - \tilde{q}_{k-1}}{h} = \tilde{v}_{k-1}, \qquad K_2^T L K_2 \tilde{v}_k = \tilde{p}_k. \tag{4}$$

System (4) (referred to as forward Euler variational integrator (VI EFD)) provides an iteration scheme that is symplectic-momentum preserving (cf. [1]). The use of different approximation rules in (3) leads to different schemes whose applicability depends on the degeneracy of the Lagrangian system. For example, the use of the backward difference operator in (3) (VI EBD) leads to an explicit update for $\tilde{p}$ and an implicit update for $\tilde{q}$ and is only applicable if the system (2) is non-degenerate. A more detailed analysis regarding the use of different discretization schemes and preservation properties is given in [3].

For the numerical computations we compare the exact solution with those obtained with the three different variational integrators (VI EBD, VI EFD both of first order and VI of second order (based on midpoint rule)), a Runge-Kutta method of fourth order (RK4), and a BDF method of second order (MNA BDF). For all methods we use a constant step size h. On edge $1 - 5$ of the circuit in Figure 1 (left) there is a pair of capacitor and inductor, on edge 6 is only one capacitor is present. Since no resistor or voltage source is involved, the energy of the system should be preserved. As shown in Figure 1 (middle), the energy is dissipating for RK and the BDF, whereas for VI it (its median, respectively) is preserved. Adding on each branch a resistor, the energy in the system decays as shown for the exact solution in Figure 1 (right). The VI solution respects this energy decay much better than RK and BDF. As second example, we consider the LC circuit given in Figure 2 (left). In Figure 2 (middle and left) the charge of the first capacitor on two time intervals is depicted. While there are no frequency damping effects for VI, the amplitude of the faster frequency is damped out for increasing integration time for RK and BDF.

References

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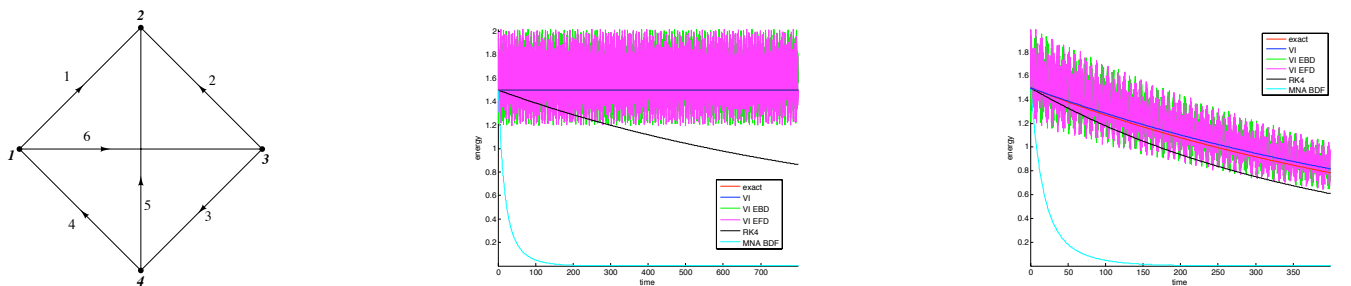
Fig. 1 Left: Circuit graph. Middle: Energy behavior for LC circuit. Right: Energy behavior for RLC circuit. Step size $h = 0.4$.
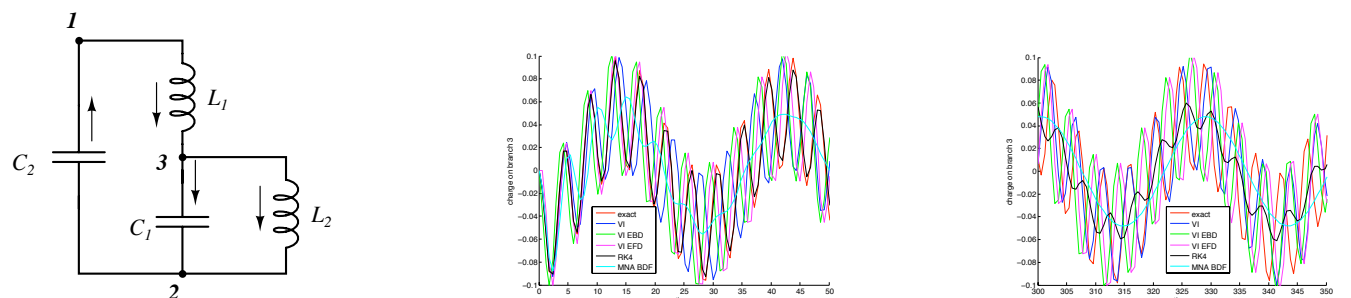

Fig. 2 Left: LC circuit. Middle and right: Charge on first capacitor for the time span $[0, 50]$ (middle) and $[300, 350]$ (right) for $h = 0.6$.