Bayesian and Game theoretical Methods for numerical PDEs

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ICERM June 6, 2017

[Gamblets for opening the complexity-bottleneck of implicit schemes for hyperbolic and parabolic ODEs/PDEs with rough coefficients, 2016. H. Owhadi and L. Zhang. arXiv:1606.07686]

[H. Owhadi and C. Scovel. Universal Scalable Robust Solvers from Computational Information Games and fast eigenspace adapted Multiresolution Analysis 2017. arXiv:1703.10761]

DARPA EQUiPS / AFOSR award no FA9550-16-1-0054
(Computational Information Games)

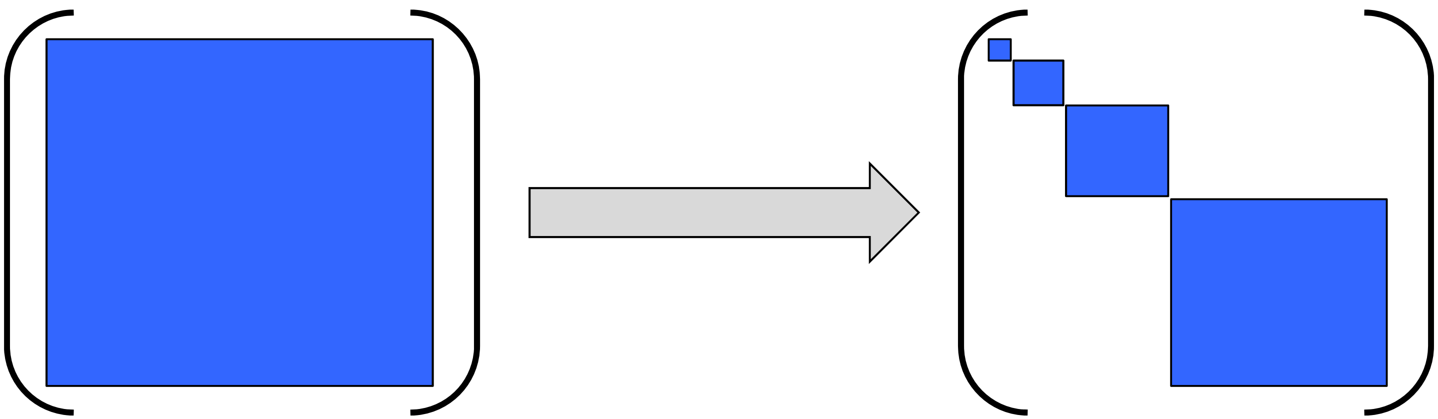
$\mathcal{L}$: Arbitrary symmetric positive continuous linear bijection

$$H_0^s(\Omega) \xrightarrow{\quad\quad\mathcal{L}\quad\quad} H^{-s}(\Omega)$$

Gamblet transform

$$H_0^s(\Omega) = \mathfrak{W}^{(1)} \oplus \mathfrak{W}^{(2)} \oplus \mathfrak{W}^{(3)} \oplus \cdots$$

$$\|u\|^2 := \int_\Omega u\mathcal{L}u$$

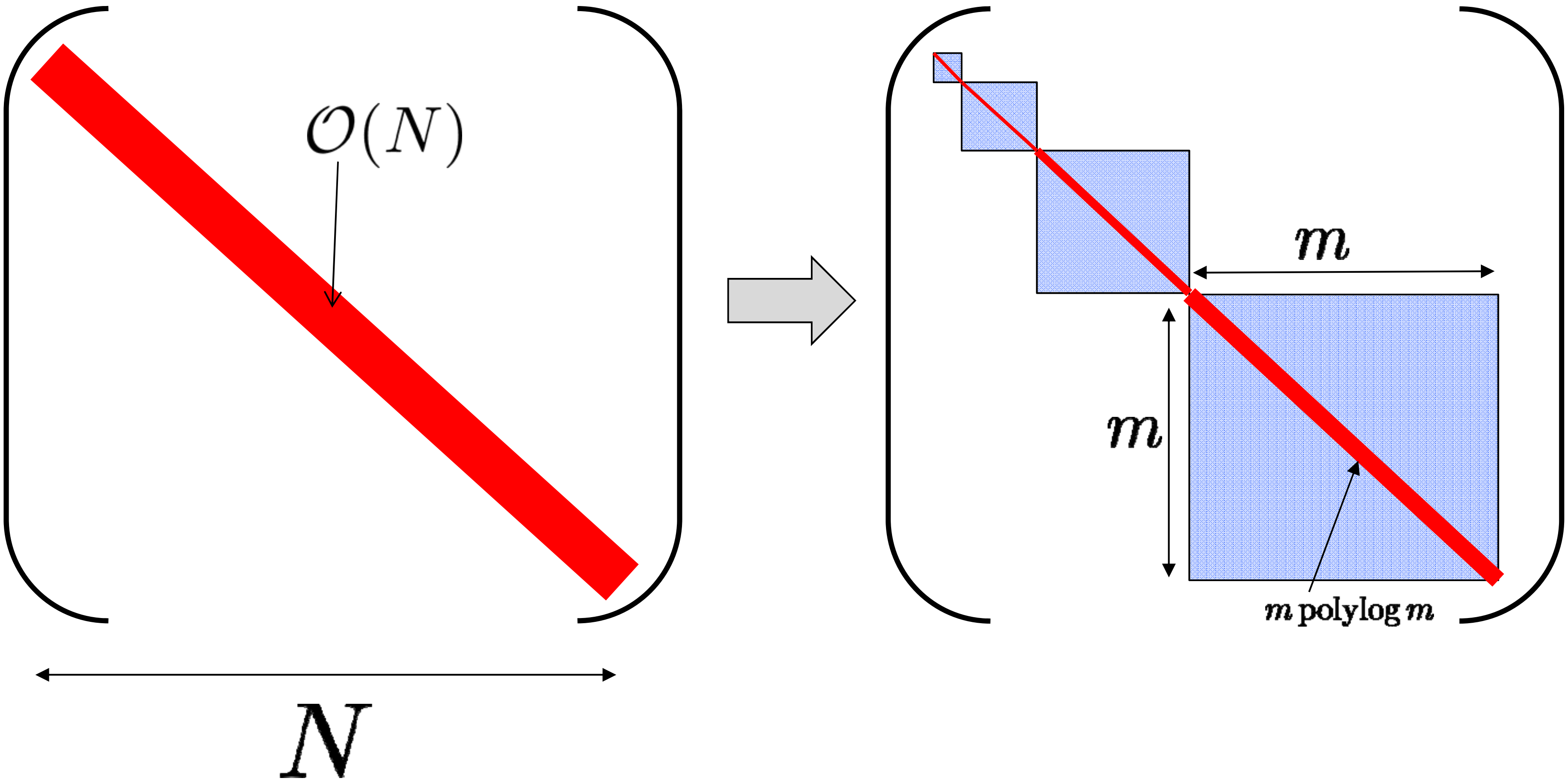


Theorem

Blocks have uniformly bounded condition numbers

Theorem If $\mathcal{L}$ is local $\langle u, v \rangle = 0$ if u and v have disjoint supports

then blocks are uniformly sparse

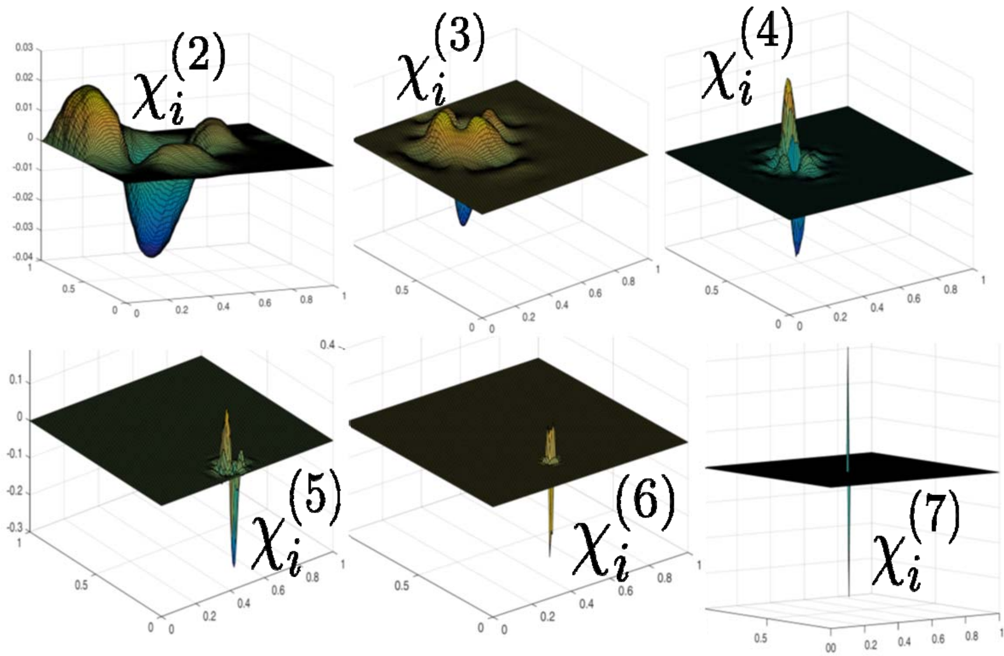


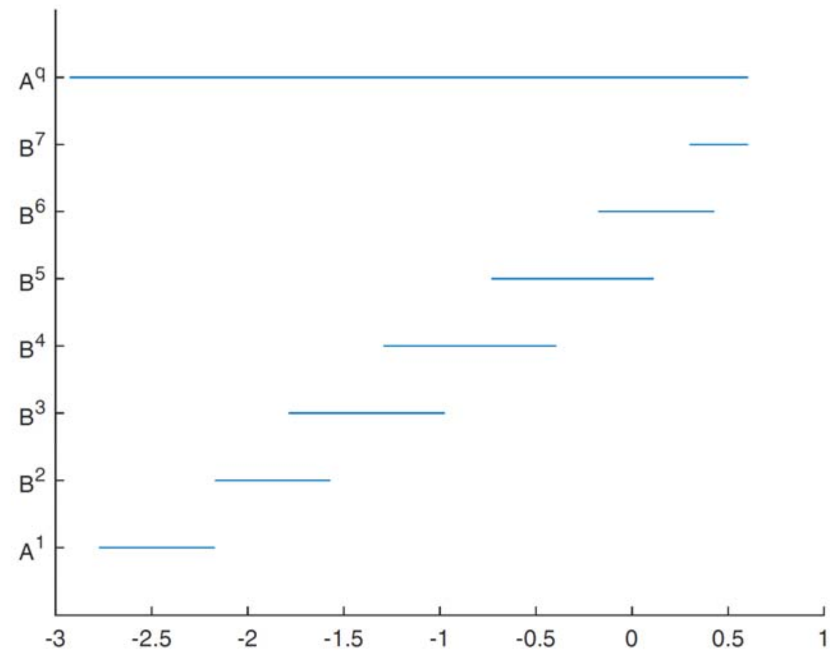
Complexity $\mathcal{O}(N \operatorname{polylog} N)$

Basis functions are operator adapted wavelets

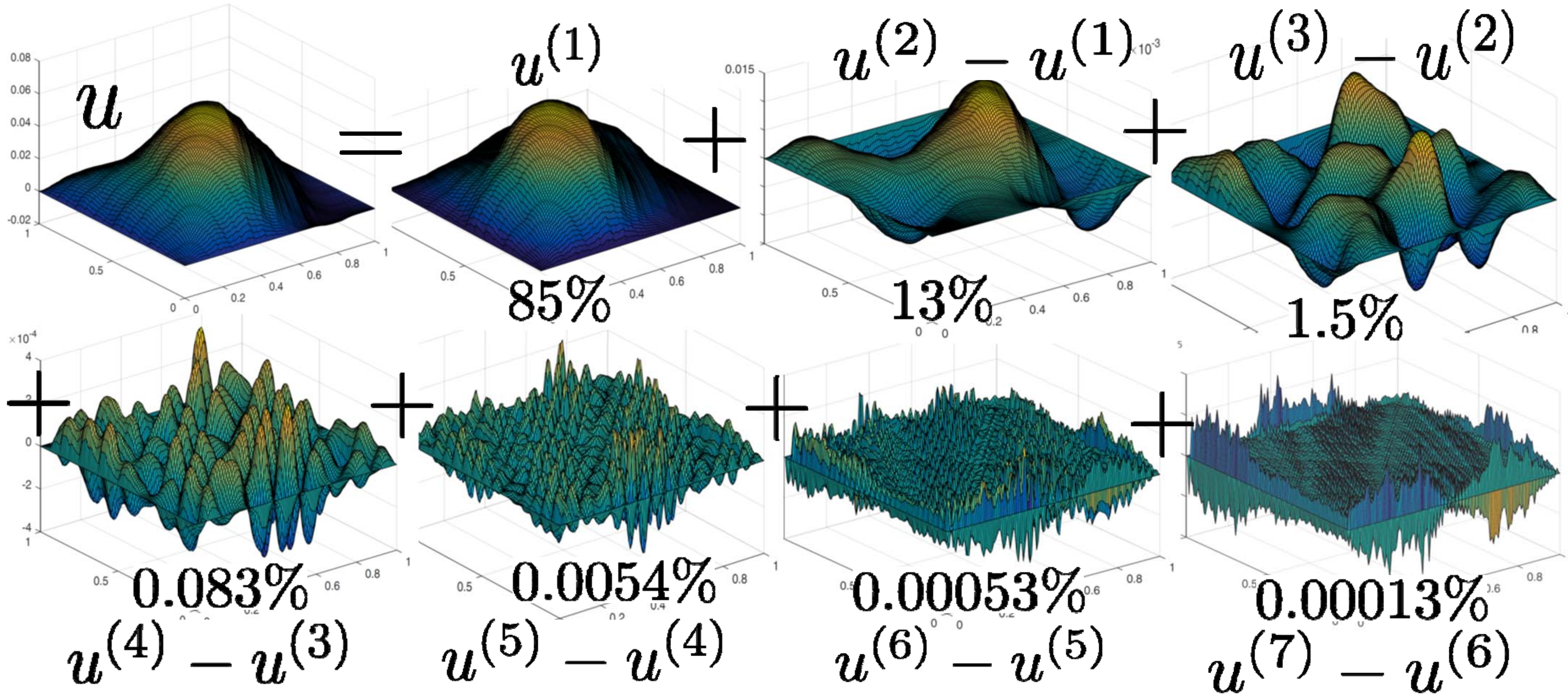
$$H_0^s(\Omega) = \mathfrak{V}^{(1)} \oplus \mathfrak{W}^{(2)} \oplus \mathfrak{W}^{(3)} \oplus \cdots$$

$$\mathfrak{W}^{(k)} := \operatorname{span}\{\chi_i^{(k)} \mid i \in \mathcal{I}^{(k)}\}$$





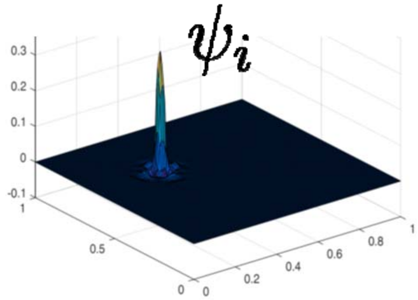
Wannier functions



$$\begin{cases} -\operatorname{div}(a\nabla u) = g, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

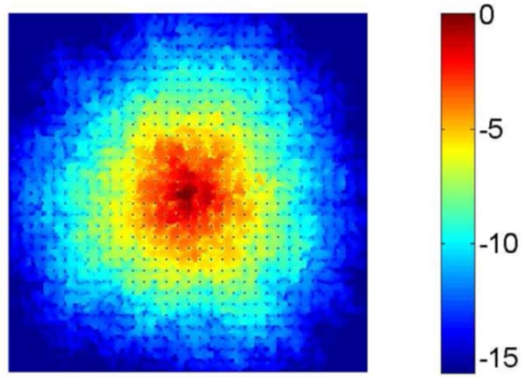
$$g \in C^{\infty}(\Omega)$$

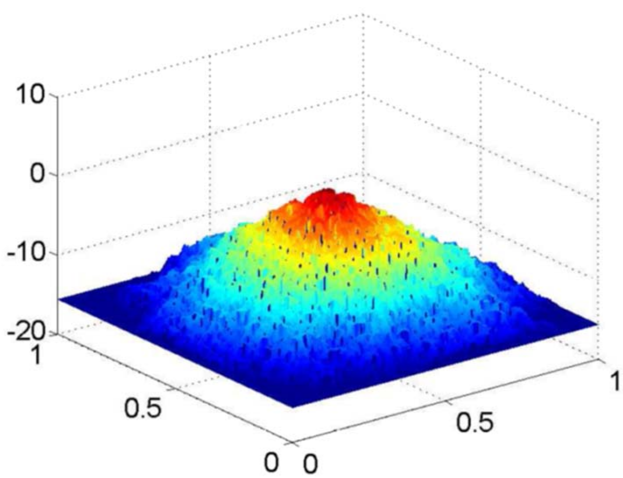
Localization of Gamblets

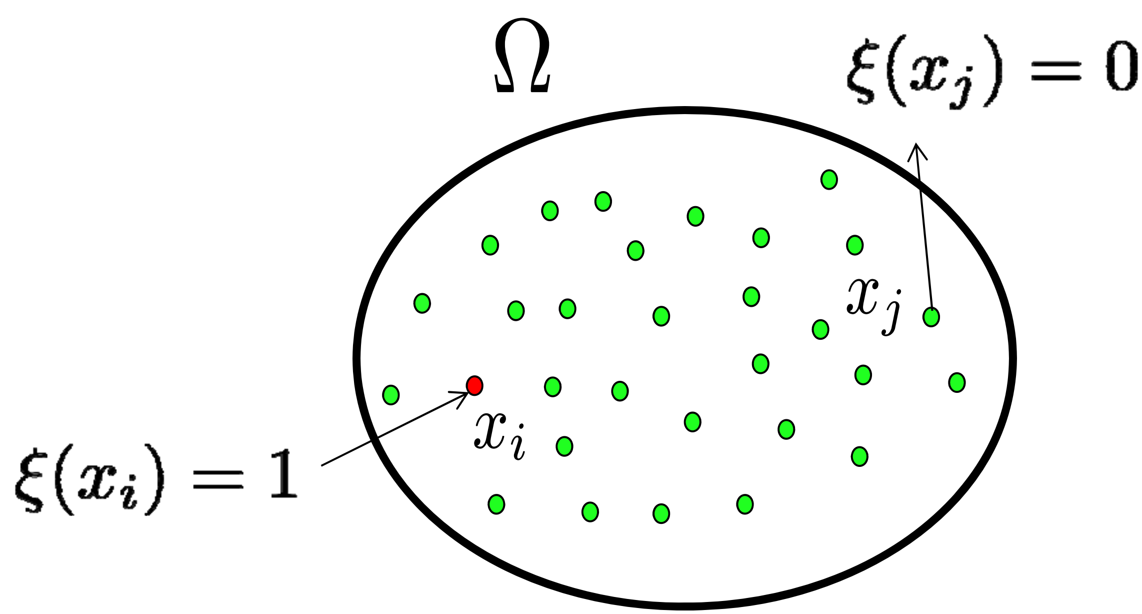


$$\xi \sim \mathcal{N}(0, \mathcal{L}^{-1})$$

$$\psi_i(x) = \mathbb{E}\big[\xi(x) \mid \xi(x_j) = \delta_{i,j} \text{ for } j \in \mathcal{I}\big]$$





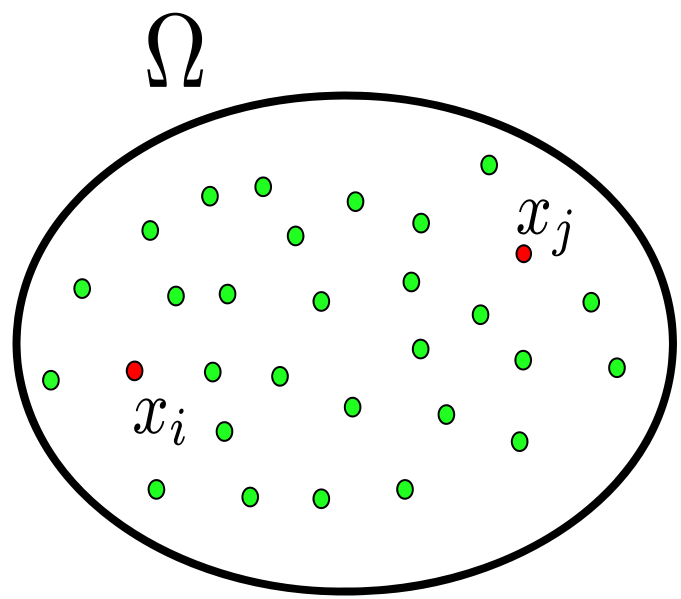


Sparsity of the precision matrix

$$\Theta_{i,j} = \mathrm{Cov}\left(\xi(x_i), \xi(x_j)\right)$$

$$\Theta^{-1}_{i,j} = \left\langle \psi_i, \psi_j \right\rangle$$

$$\Theta^{-1}_{i,j} = 0$$



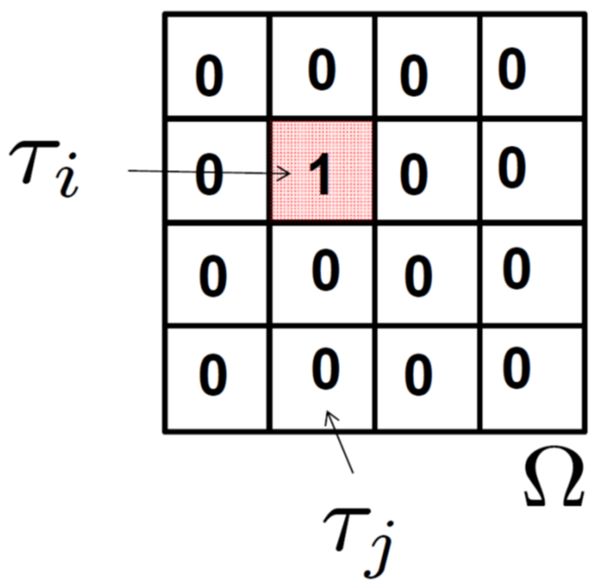
$$\Updownarrow$$

$$\mathrm{Cov}\left(\xi(x_i), \xi(x_j) \middle| \xi(x_l), l \neq i, j]\right) = 0$$

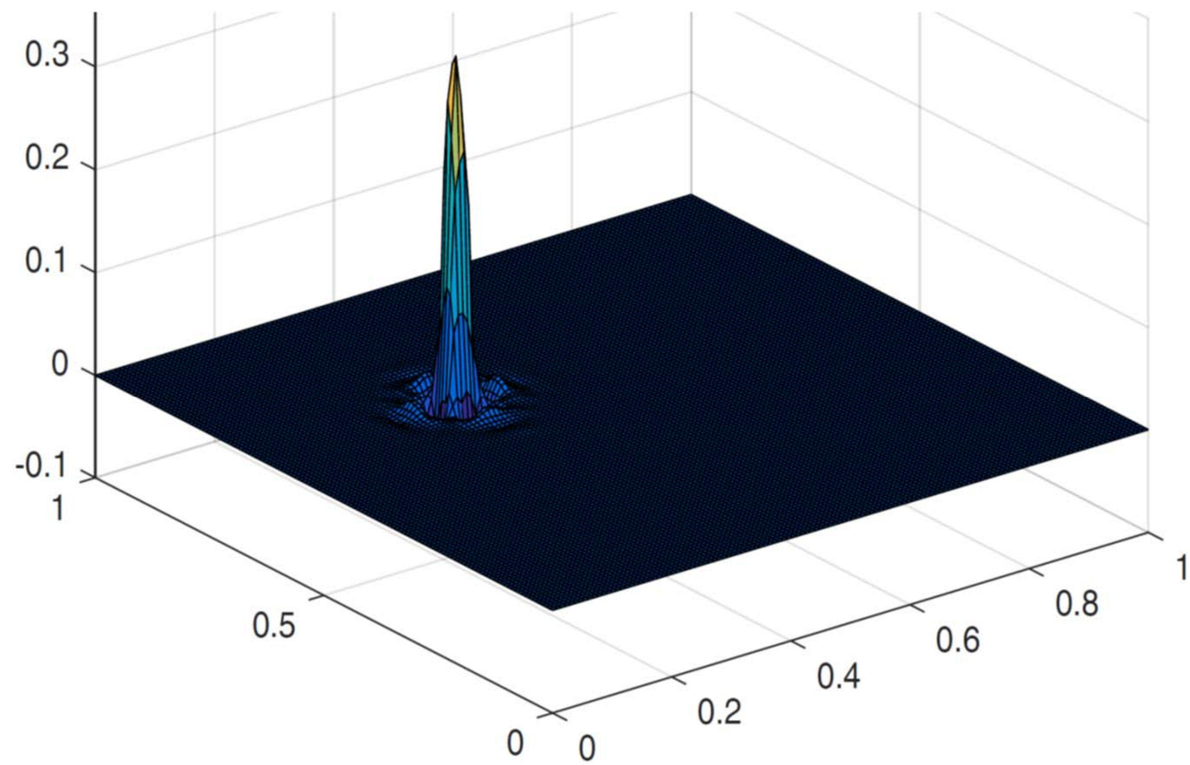
Localization of Gamblets

$$\xi \sim \mathcal{N}(0, \mathcal{L}^{-1})$$

$$\psi_i = \mathbb{E}\big[\xi \mid [\phi_j, \xi] = \delta_{i,j} \text{ for } j \in \mathcal{I}\big]$$



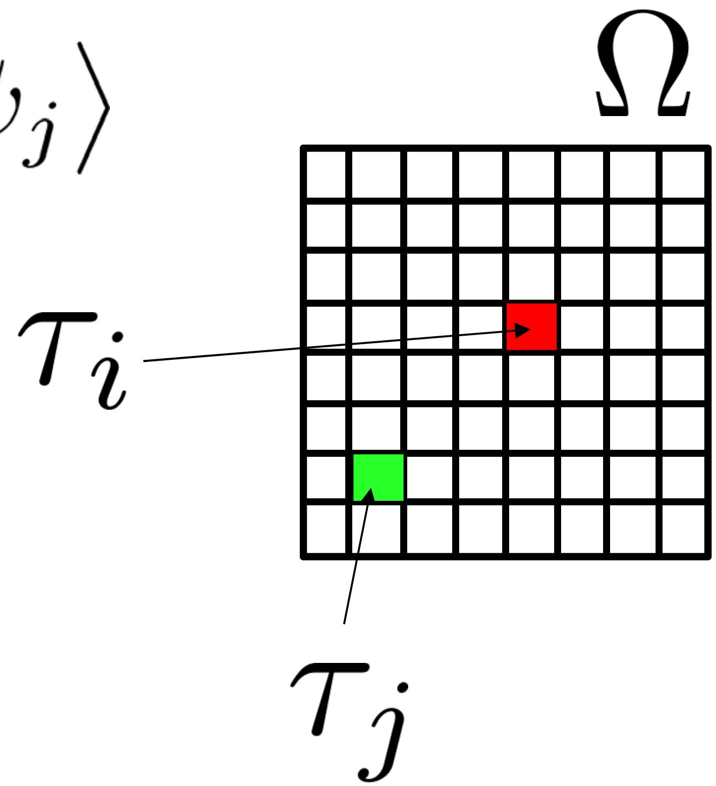
$$\phi_i = 1_{\tau_i}$$



Sparsity of the precision matrix

$$\Theta_{i,j} = \mathrm{Cov}\left([\phi_i, \xi], [\phi_j, \xi]\right)$$

$$\Theta^{-1}_{i,j} = \langle \psi_i, \psi_j \rangle$$



$$\Theta^{-1}_{i,j} = 0$$

$$\mathrm{Cov}\left([\phi_i, \xi], [\phi_j, \xi] \middle| [\phi_l, \xi], l \neq i, j]\right) = 0$$

Localization problem in Numerical Homogenization

[Chu-Graham-Hou-2010] (limited inclusions)

[Efendiev-Galvis-Wu-2010] (limited inclusions or mask)

[Babuska-Lipton 2010] (local boundary eigenvectors)

[Owhadi-Zhang 2011] (localized transfer property)

[Malqvist-Peterseim 2012] Local Orthogonal Decomposition

[Owhadi-Zhang-Berlyand 2013] (Rough Polyharmonic Splines)

[A. Gloria, S. Neukamm, and F. Otto, 2015] (quantification of ergodicity)

[Hou and Liu,DCDS-A, 2016] [Chung-Efendiev-Hou, JCP 2016]

[Owhadi, Multiresolution operator decomposition, SIREV 2017]

[Owhadi, Zhang, gamblets for hyperbolic and parabolic PDEs, 2016]

[Hou, Qin, Zhang, 2016] [Hou, Zhang, 2017]

[Hou and Zhang, 2017]: Higher order PDEs (localization under strong ellipticity, h sufficiently small, and higher order polynomials as measurement functions)

[Kornhuber, Peterseim, Yserentant, 2016]: Subspace decomposition

Subspace decomposition/correction and Schwarz iterative methods

[J. Xu, 1992]: Iterative methods by space decomposition and subspace correction

[Griebel-Oswald, 1995]: Schwarz algorithms

Example

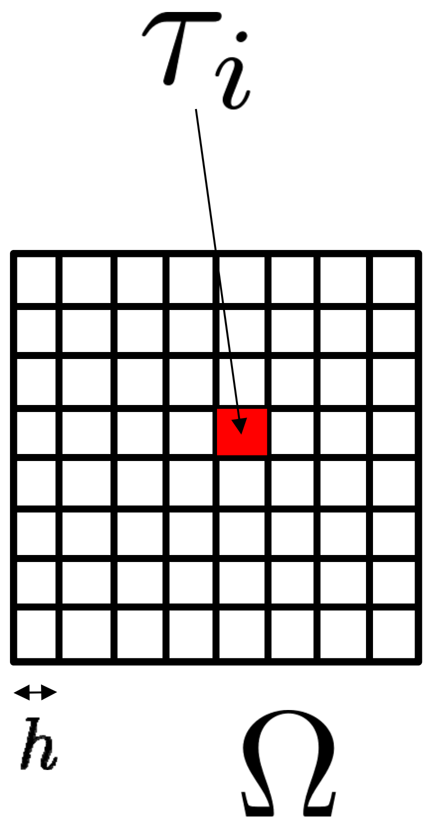
$$\mathcal{B} := H_0^s(\Omega) \qquad \|u\|^2 := [\mathcal{L}u, u]$$

$\mathcal{L}$: arbitrary continuous positive symmetric linear bijection

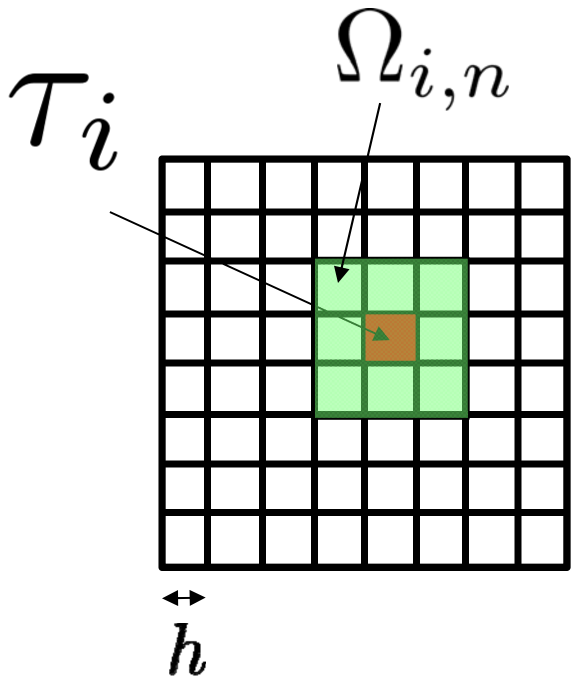
$$(H_0^s(\Omega), \|\cdot\|_{H_0^s(\Omega)}) \xrightarrow{\quad \mathcal{L} \quad} (H^{-s}(\Omega), \|\cdot\|_{H^{-s}(\Omega)})$$

$\mathcal{L}$ is local $\langle u, v\rangle = 0$ if u and v have disjoint supports

Examples



- $\phi_i = \frac{1_{\tau_i}}{\sqrt{|\tau_i|}}.$
- $\phi_i = \delta(\cdot - x_i),$
  $(s > \frac{d}{2})$
- $(\phi_{i,\alpha})_{\alpha \in \beth}$
  forms an orthonormal basis of $\mathcal{P}_{s-1}(\tau_i)$

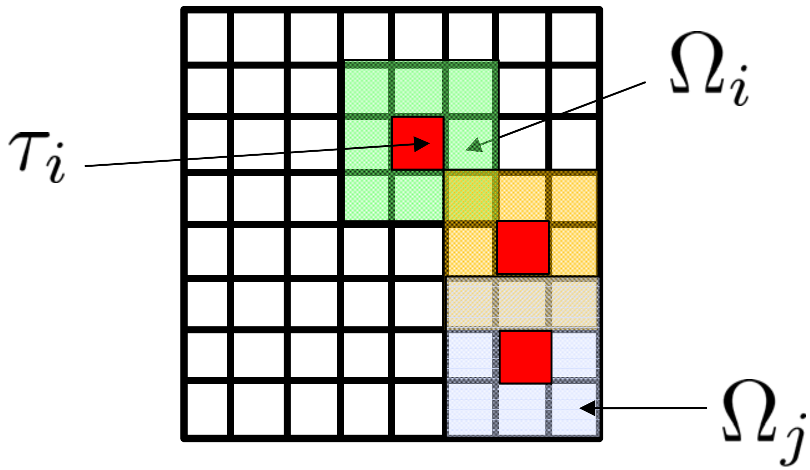


$$\operatorname{dist}(\tau_i, \partial\Omega_{i,n}) \approx nh$$

$\psi_{i,\alpha}^n$: Localization of $\psi_{i,\alpha}$ to $\Omega_{i,n}$

Theorem

$$\|\psi_{i,\alpha} - \psi_{i,\alpha}^n\|_{H_0^s(\Omega)} \leq Ce^{-n/C}$$



$$\Omega = \cup_i \Omega_i$$

$$H_0^s(\Omega) = \sum_{i \in \beth} H_0^s(\Omega_i)$$

Condition for localization

For $\varphi \in H^{-s}(\Omega)$

$$C_{\min} \leq \frac{\sum_i \inf_{\phi \in \Phi} \|\varphi - \phi\|^2_{H^{-s}(\Omega_i)}}{\inf_{\phi \in \Phi} \|\varphi - \phi\|^2_{H^{-s}(\Omega)}} \leq C_{\max}$$

$$\Phi = \{\phi_{i,\alpha} \mid (i,\alpha) \in \beth \times \aleph\}$$

Theorem

Assume that there exists a constant C_0 such that $|\aleph| \leq C_0$,

- $\|D^t f\|_{L^2(\Omega)} \leq C_0 h^{s-t} \|f\|_{H_0^s(\Omega)}$ for $t \in \{0, 1, \ldots, s\}$, for $f \in H_0^s(\Omega)$ such that $[\phi_{i,\alpha}, f] = 0$ for $(i, \alpha) \in \beth \times \aleph$,
- $\sum_{i \in \beth, \alpha \in \aleph} [\phi_{i,\alpha}, f]^2 \leq C_0 \big(\|f\|^2_{L^2(\Omega)} + h^{2s} \|f\|^2_{H_0^s(\Omega)} \big)$, for $f \in H_0^s(\Omega)$, and
- $|x|^2 \leq C_0 h^{-2s} \| \sum_{\alpha \in \aleph} x_\alpha \phi_{i,\alpha} \|^2_{H^{-s}(\tau_i)}$, for $i \in \beth$ and $x \in \mathbb{R}^\aleph$.

Then for $\varphi \in H^{-s}(\Omega)$

$$C_{\min} \leq \frac{\sum_i \inf_{\phi \in \Phi} \|\varphi - \phi\|^2_{H^{-s}(\Omega_i)}}{\inf_{\phi \in \Phi} \|\varphi - \phi\|^2_{H^{-s}(\Omega)}} \leq C_{\max}$$

Where $C_{\max}, C_{\min}$ depend only on C_0, d, δ and s

Banach space setting

$$\mathcal{B} = \sum_{i \in \beth} \mathcal{B}_i$$

$\|\cdot\|_i$ and $\|\cdot\|_{i,*}$ norms induced by $\|\cdot\|$ on $\mathcal{B}_i$ and $\mathcal{B}_i^*$

Condition for localization

For $\varphi \in \mathcal{B}^*$

$$C_{\min} \leq \frac{\sum_i \inf_{\phi \in \Phi} \|\varphi - \phi\|_{i,*}^2}{\inf_{\phi \in \Phi} \|\varphi - \phi\|_*^2} \leq C_{\max}$$

$$\Phi = \{\phi_{i,\alpha} \mid (i,\alpha) \in \beth \times \aleph\}$$

Operator connectivity distance

C: $\beth \times \beth$ connectivity matrix

$C_{i,j} = 1$ if $\exists (\chi_i, \chi_j) \in \mathcal{B}_i \times \mathcal{B}_j$ s.t. $\langle \chi_i, \chi_j \rangle \neq 0$

$C_{i,j} = 0$ otherwise

$\mathbf{d}$: Graph distance on $\beth$ induced by C

$\psi_{i,\alpha}^n$: Localization of $\psi_{i,\alpha}$ to $\mathcal{B}_i^n$

$$\mathcal{B}_i^n = \cup_{j:\mathbf{d}(i,j)\leq n} B_i$$

Theorem Under localization conditions

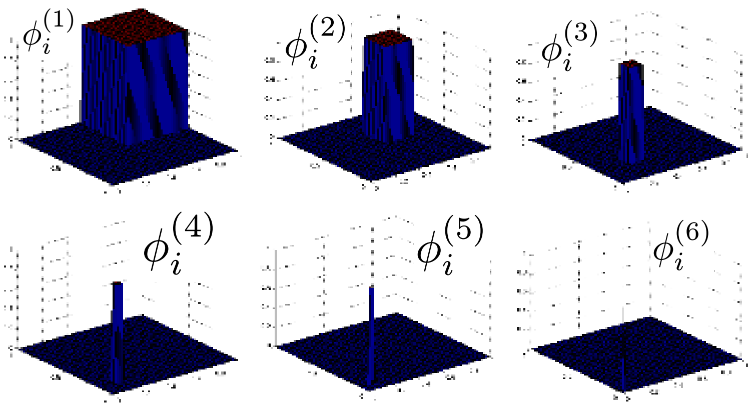
$$\|\psi_{i,\alpha} - \psi_{i,\alpha}^n\| \leq C e^{-n/C}$$

$$H_0^s(\Omega) \xrightarrow{\quad\mathcal{L}\quad} H^{-s}(\Omega)$$

$$\cup$$

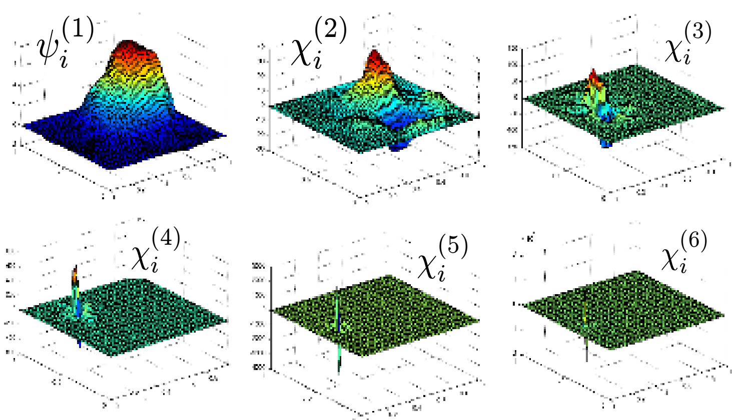
$$L^2(\Omega)$$

The method



Haar-wavelet decomposition of $L^2(\Omega) \rightarrow H^{-s}(\Omega)$

Gamblet transform



Multi-resolution decomposition of $H_0^s(\Omega) \rightarrow H^{-s}(\Omega)$

Gamblet Transform/Solve

For $i \in \mathcal{I}^{(q)}$, $\psi_i^{(q)} = \varphi_i$

For $i \in \mathcal{I}^{(q)}$, $g_i^{(q)} = [g, \psi_i^{(q)}]$

For $i, j \in \mathcal{I}^{(q)}$, $A_{i,j}^{(q)} = \langle \psi_i^{(q)}, \psi_j^{(q)} \rangle$

for $k = q$ to 2 **do**

$B^{(k)} = W^{(k)} A^{(k)} W^{(k),T}$

$w^{(k)} = B^{(k),-1} W^{(k)} g^{(k)}$

For $i \in \mathcal{J}^{(k)}$, $\chi_i^{(k)} = \sum_{j \in \mathcal{I}^{(k)}} W_{i,j}^{(k)} \psi_j^{(k)}$

$u^{(k)} - u^{(k-1)} = \sum_{i \in \mathcal{J}^{(k)}} w_i^{(k)} \chi_i^{(k)}$

$D^{(k,k-1)} = -B^{(k),-1} W^{(k)} A^{(k)} \bar{\pi}^{(k,k-1)}$

$R^{(k-1,k)} = \bar{\pi}^{(k-1,k)} + D^{(k-1,k)} W^{(k)}$

$A^{(k-1)} = R^{(k-1,k)} A^{(k)} R^{(k,k-1)}$

For $i \in \mathcal{I}^{(k-1)}$, $\psi_i^{(k-1)} = \sum_{j \in \mathcal{I}^{(k)}} R_{i,j}^{(k-1,k)} \psi_j^{(k)}$

$g^{(k-1)} = R^{(k-1,k)} g^{(k)}$

end for

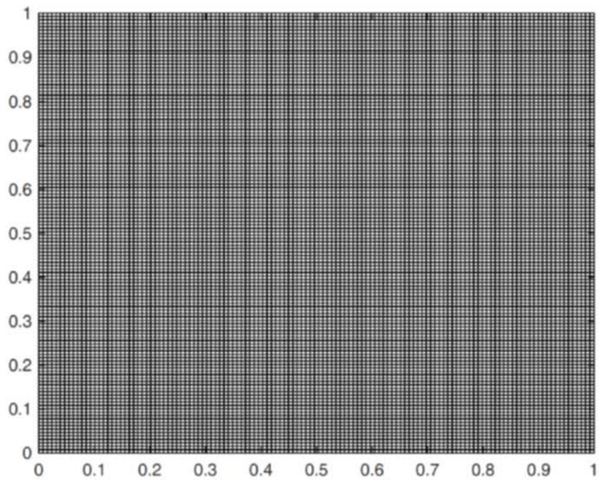
$U^{(1)} = A^{(1),-1} g^{(1)}$

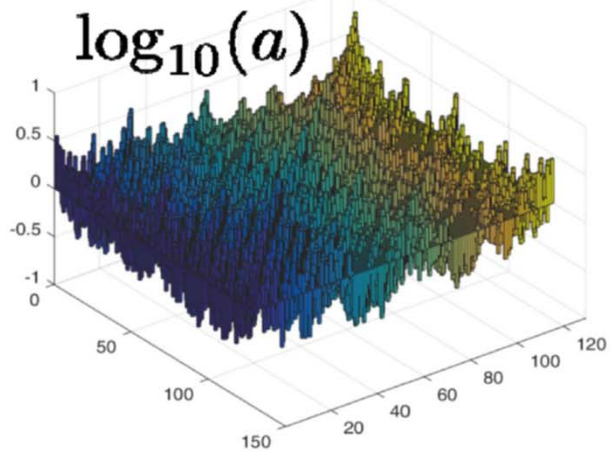
$u^{(1)} = \sum_{i \in \mathcal{I}^{(1)}} U_i^{(1)} \psi_i^{(1)}$

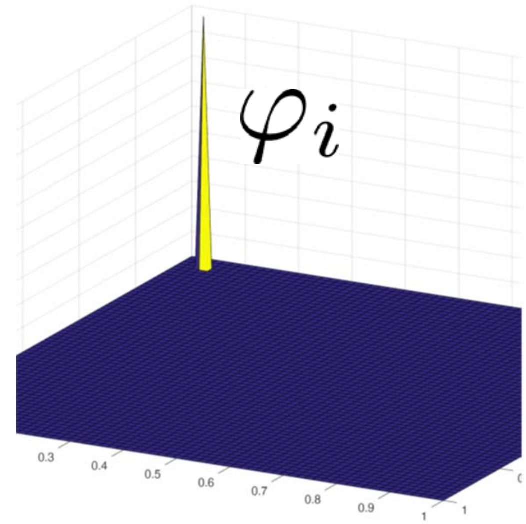
$u = u^{(1)} + (u^{(2)} - u^{(1)}) + \cdots + (u^{(q)} - u^{(q-1)})$

Example

$$\begin{cases} -\operatorname{div}(a\nabla u) = g, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$





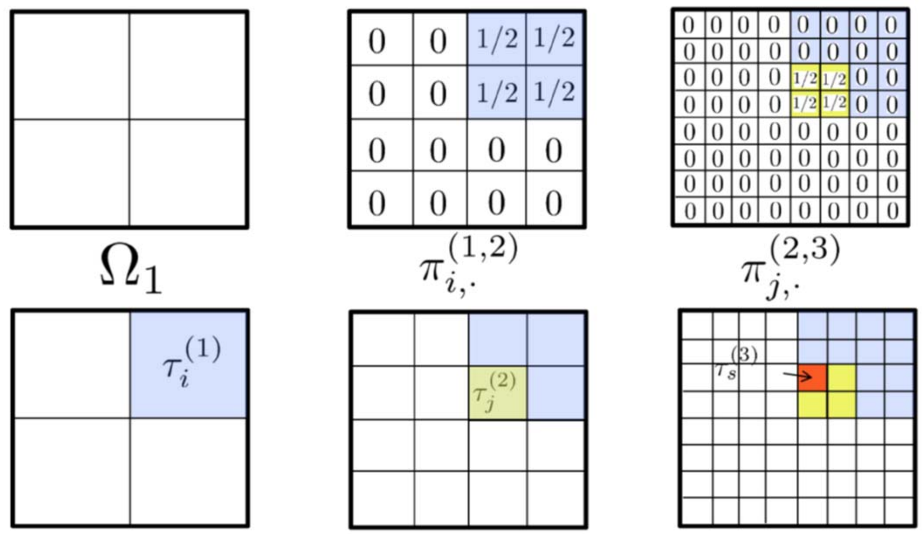


$$\mathcal{B} = \{\varphi_i | i \in \mathcal{I}\} \subset H_0^1(\Omega)$$

$$\|u\|^2 := \int_\Omega (\nabla u)^T a \nabla u$$

Inputs of the algorithm

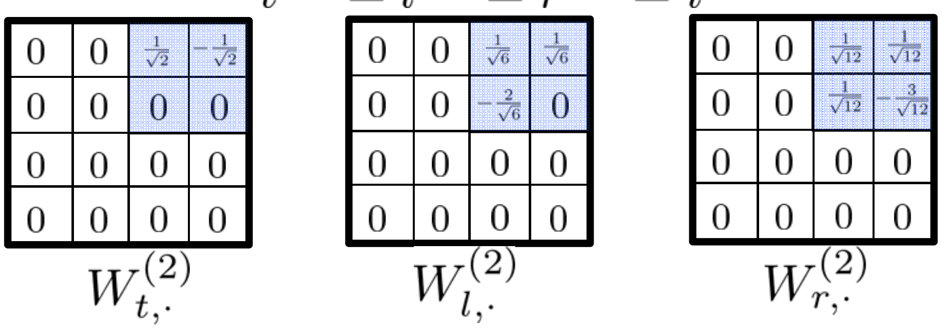
$$A_{i,j} = \int_\Omega (\nabla \varphi_i)^T a \nabla \varphi_j$$



$$\pi^{(k-1,k)}$$

$$\pi^{(k-1,k)}(\pi^{(k-1,k)})^T = I^{(k-1)}$$

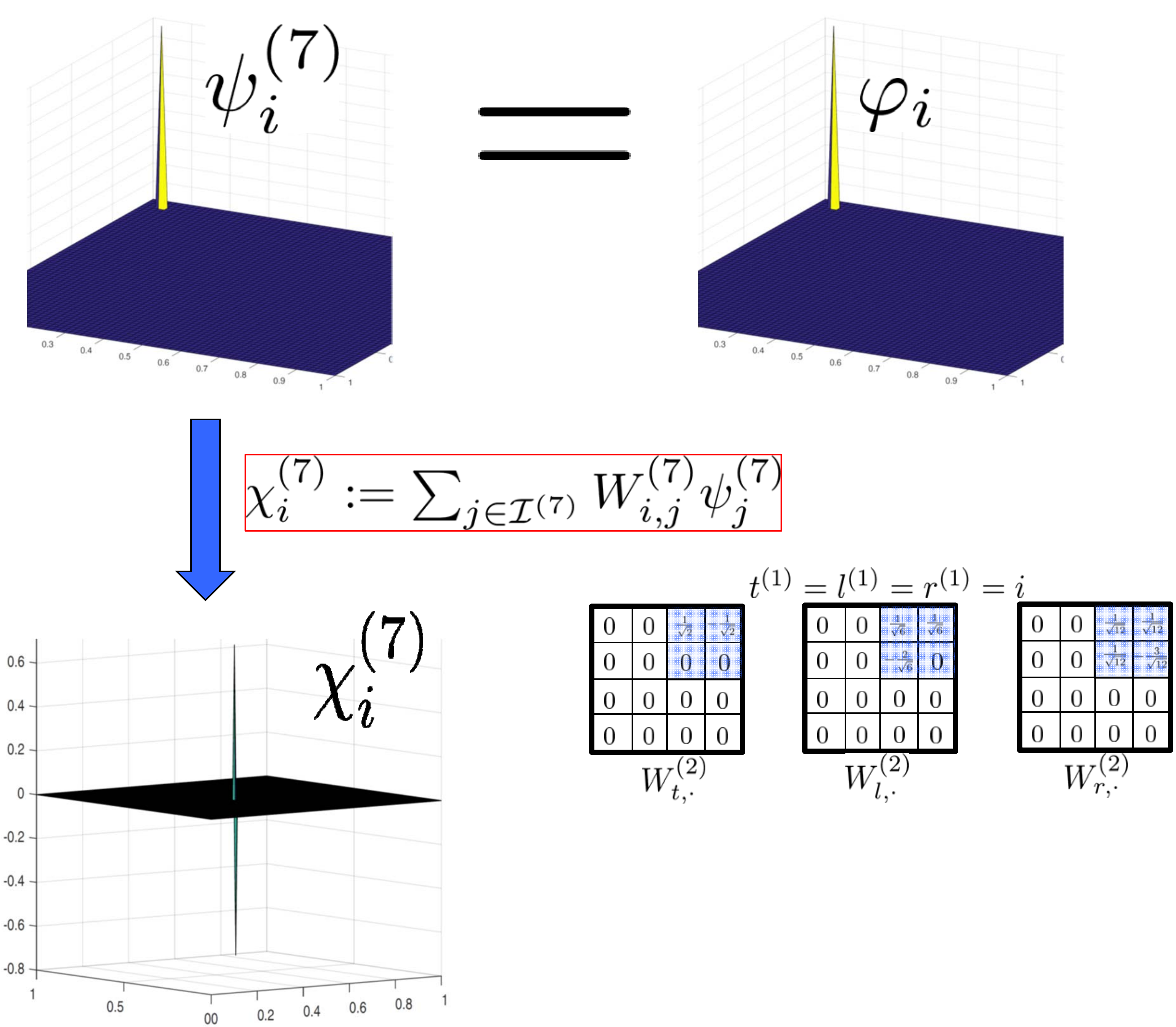
$$t^{(1)} = l^{(1)} = r^{(1)} = i$$



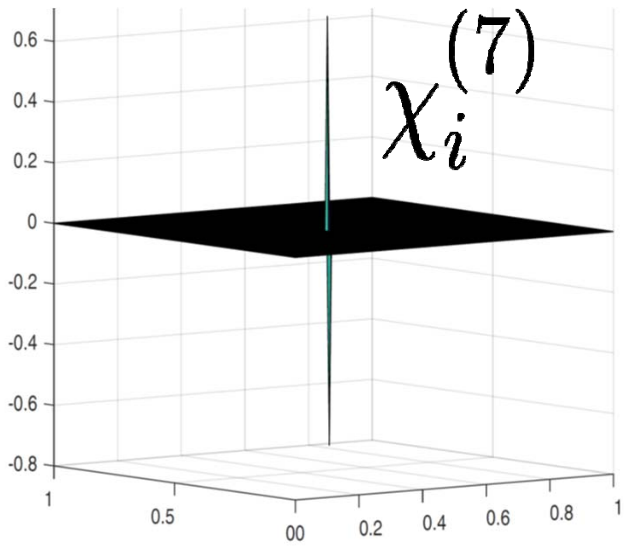
$$W^{(k)}:$$

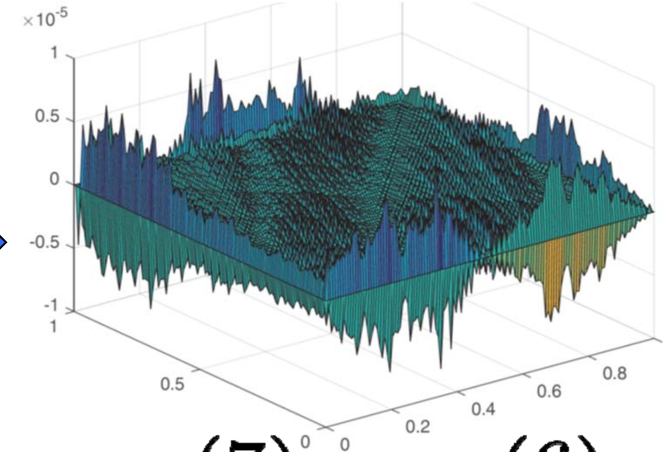
$$\mathrm{Img}(W^{(k),T}) = \mathrm{Ker}(\pi^{(k-1,k)})$$

$$W^{(k),T} W^{(k)} = J^{(k)}$$



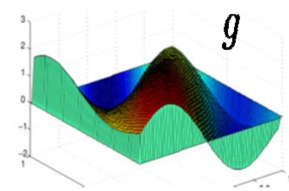
$\psi_i^{(7)}$
$=$
φ_i
$\chi_i^{(7)} := \sum_{j \in \mathcal{I}^{(7)}} W_{i,j}^{(7)} \psi_j^{(7)}$
$\chi_i^{(7)}$
$t^{(1)} = l^{(1)} = r^{(1)} = i$
$W_{t,\cdot}^{(2)}$
$W_{l,\cdot}^{(2)}$
$W_{r,\cdot}^{(2)}$





$$u^{(7)} - u^{(6)}$$

$$u^{(7)} - u^{(6)} = \sum_i w_i^{(7)} \chi_i^{(7)}$$

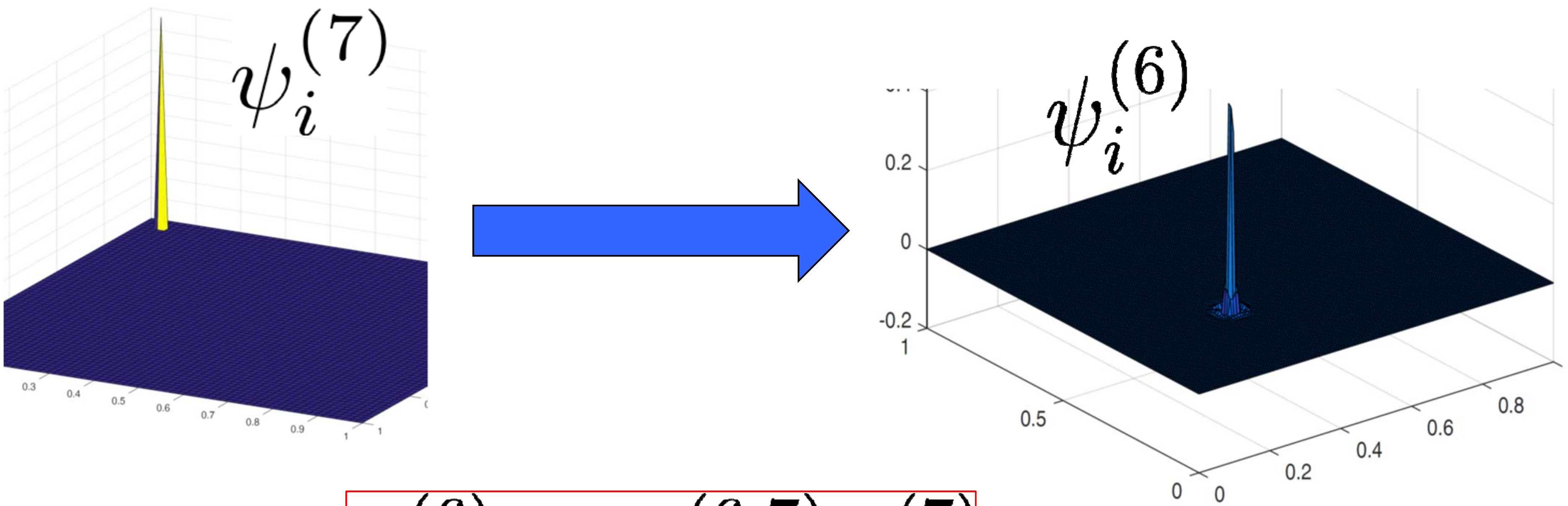


$$B^{(7)} w^{(7)} = W^{(7),T} g^{(7)}$$

$$g_i^{(7)} = \int_\Omega g \psi_i^{(7)}$$

$$B_{i,j}^{(7)} = \int_\Omega (\nabla \chi_i^{(7)})^T a \nabla \chi_j^{(7)}$$

$$B^{(7)} = W^{(7)} A W^{(7),T}$$

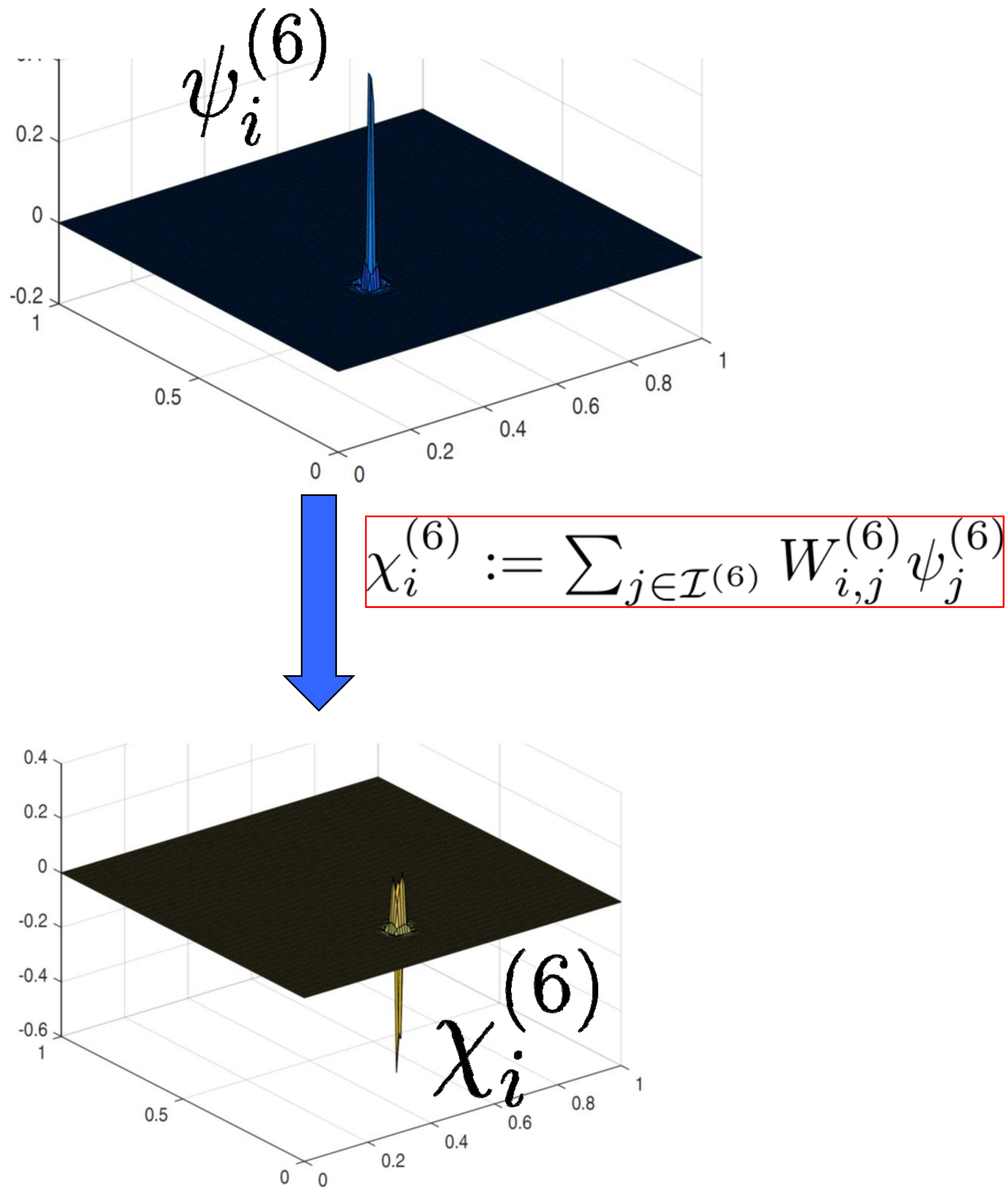


$$\psi_i^{(6)} = R_{i,j}^{(6,7)} \psi_j^{(7)}$$

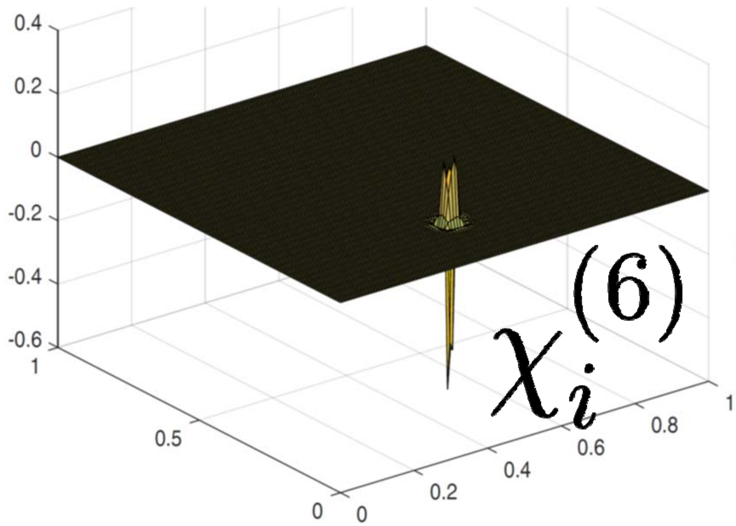
$$A^{(7)} = A$$

$$R^{(6,7)} = \pi^{(6,7)} (I^{(7)} - A^{(7)} W^{(7),T} B^{(7),-1} W^{(7)})$$

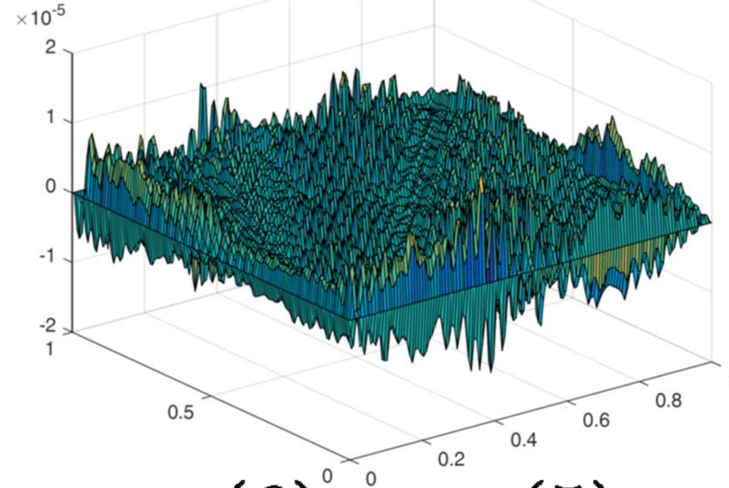
$$A^{(6)} = R^{(6,7)} A^{(7)} (R^{(6,7)})^T$$



$\psi_i^{(6)}$
0.2
0
-0.2
1
0.5
0
0
0.2
0.4
0.6
0.8
1
$\chi_i^{(6)} := \sum_{j\in\mathcal{I}^{(6)}} W_{i,j}^{(6)}\psi_j^{(6)}$
0.4
0.2
0
-0.2
-0.4
-0.6
1
0.5
0
0
0.2
0.4
0.6
0.8
1
$\chi_i^{(6)}$



$$\chi_i^{(6)}$$

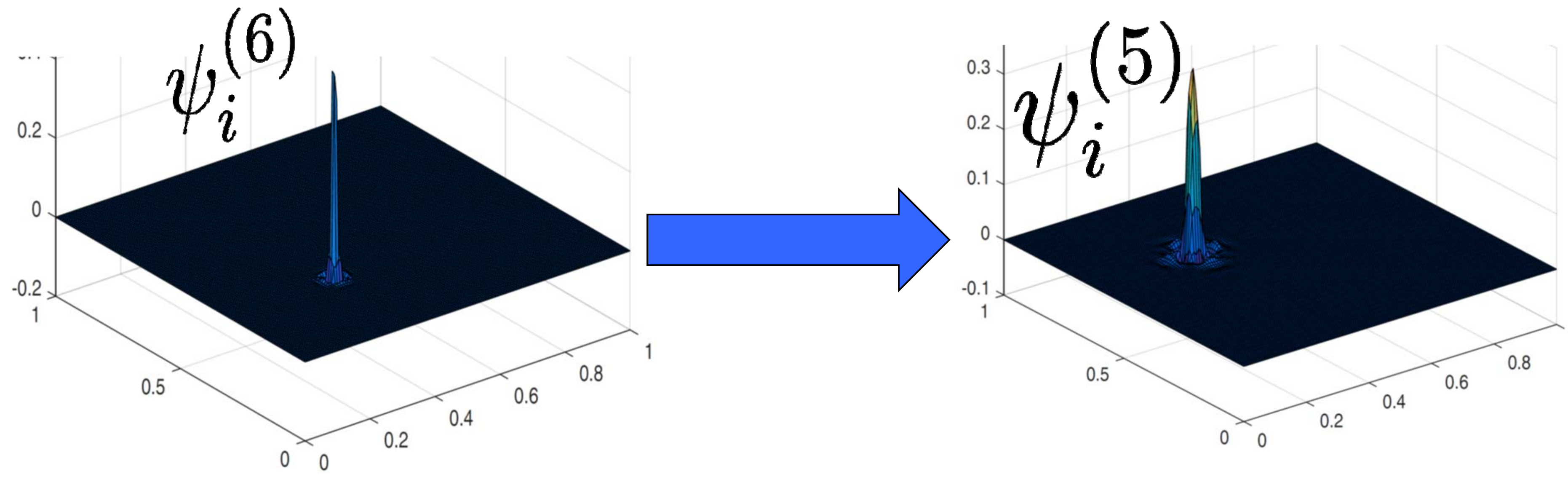


$$u^{(6)} - u^{(5)}$$

$$u^{(6)} - u^{(5)} = \sum_i w_i^{(6)} \chi_i^{(6)}$$

$$B^{(6)} w^{(6)} = W^{(6),T} g^{(6)} \qquad g^{(6)} = R^{(6,7)} g^{(7)}$$

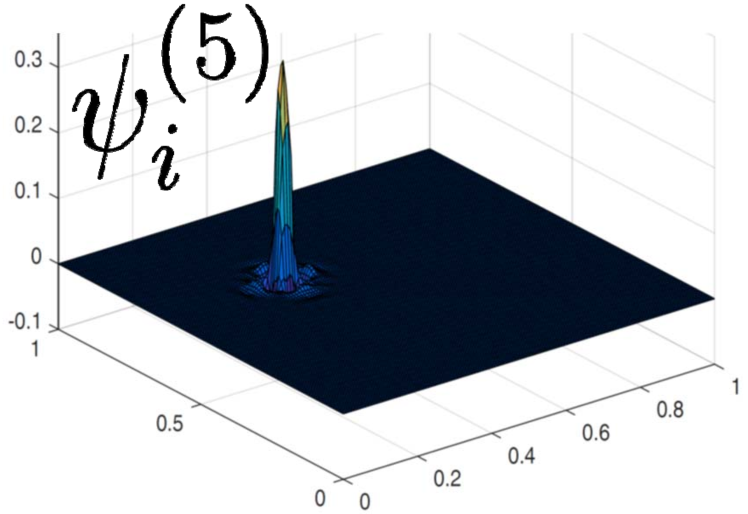
$$B^{(6)} = W^{(6)} A^{(6)} W^{(6),T}$$



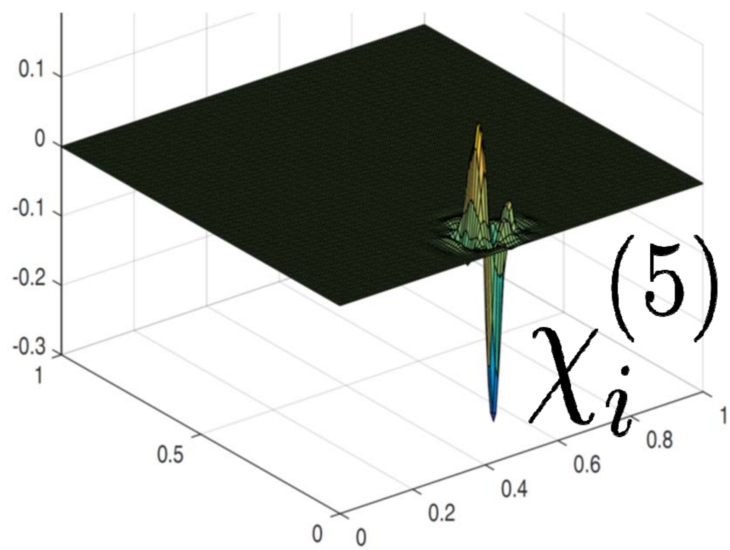
$$\psi_i^{(5)} = R_{i,j}^{(5,6)} \psi_j^{(6)}$$

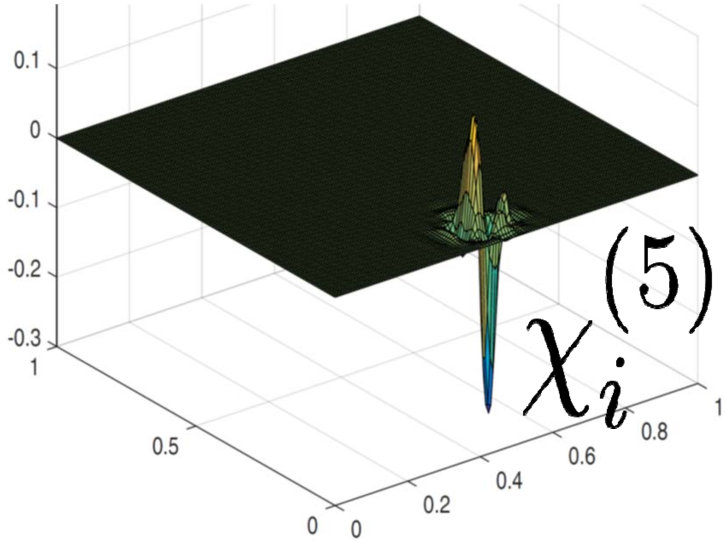
$$R^{(5,6)} = \pi^{(5,6)} (I^{(6)} - A^{(6)} W^{(6),T} B^{(6),-1} W^{(6)})$$

$$A^{(5)} = R^{(5,6)} A^{(6)} (R^{(5,6)})^T$$

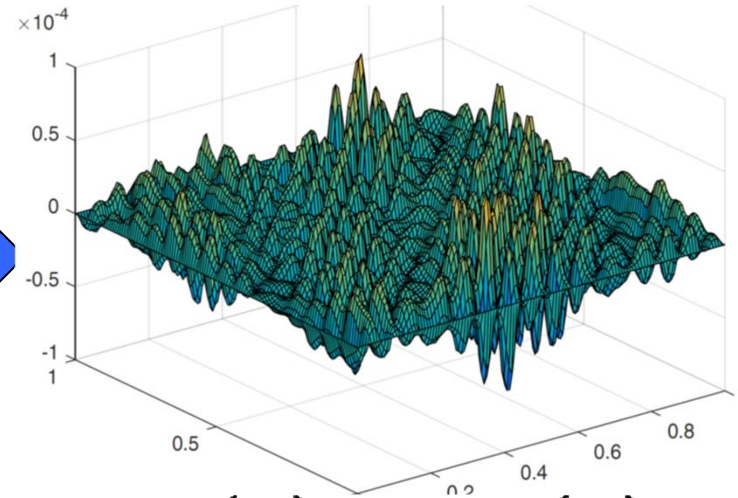


$$\chi_i^{(5)} := \sum_{j\in\mathcal{I}^{(5)}} W_{i,j}^{(5)}\,\psi_j^{(5)}$$





$$\chi_i^{(5)}$$

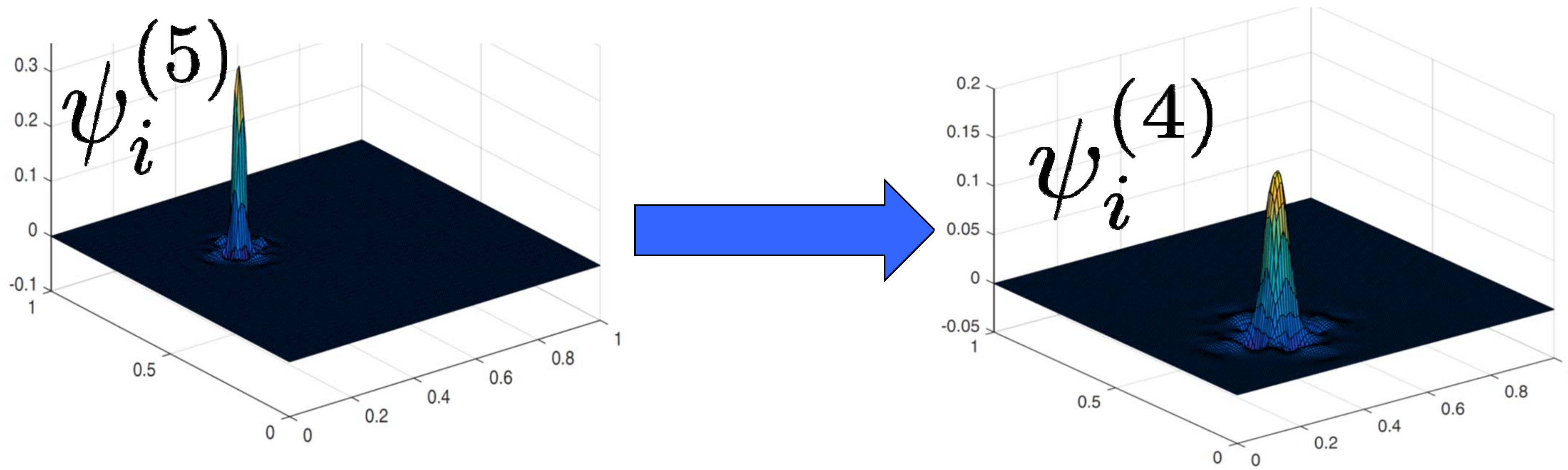


$$u^{(5)} - u^{(4)}$$

$$u^{(5)} - u^{(4)} = \sum_i w_i^{(5)} \chi_i^{(5)}$$

$$B^{(5)} w^{(5)} = W^{(5),T} g^{(5)} \qquad g^{(5)} = R^{(5,6)} g^{(6)}$$

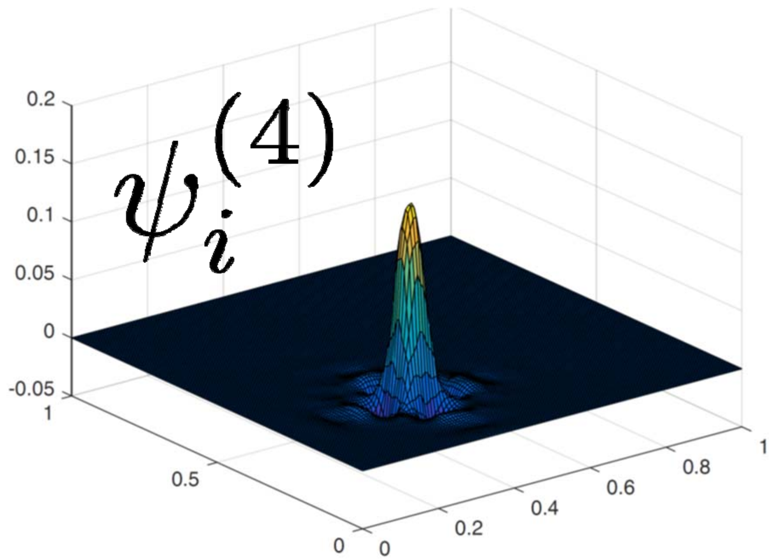
$$B^{(5)} = W^{(5)} A^{(5)} W^{(5),T}$$



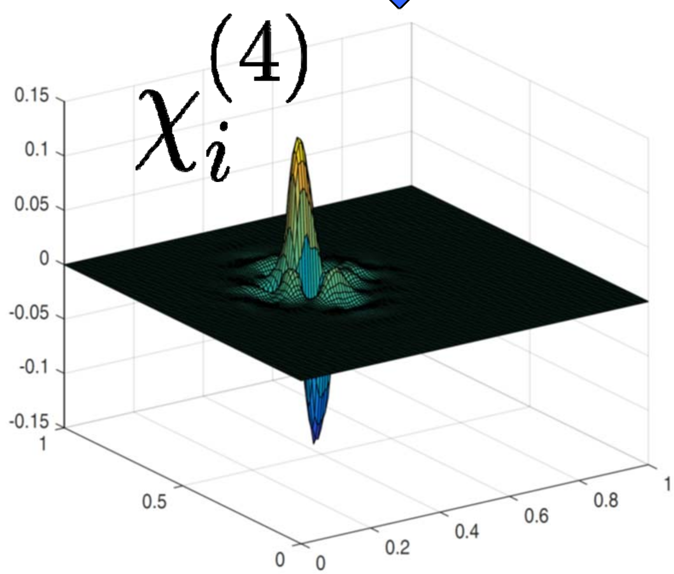
$$\psi_i^{(4)} = R_{i,j}^{(4,5)} \psi_j^{(5)}$$

$$R^{(4,5)} = \pi^{(4,5)} (I^{(5)} - A^{(5)} W^{(5),T} B^{(5),-1} W^{(5)})$$

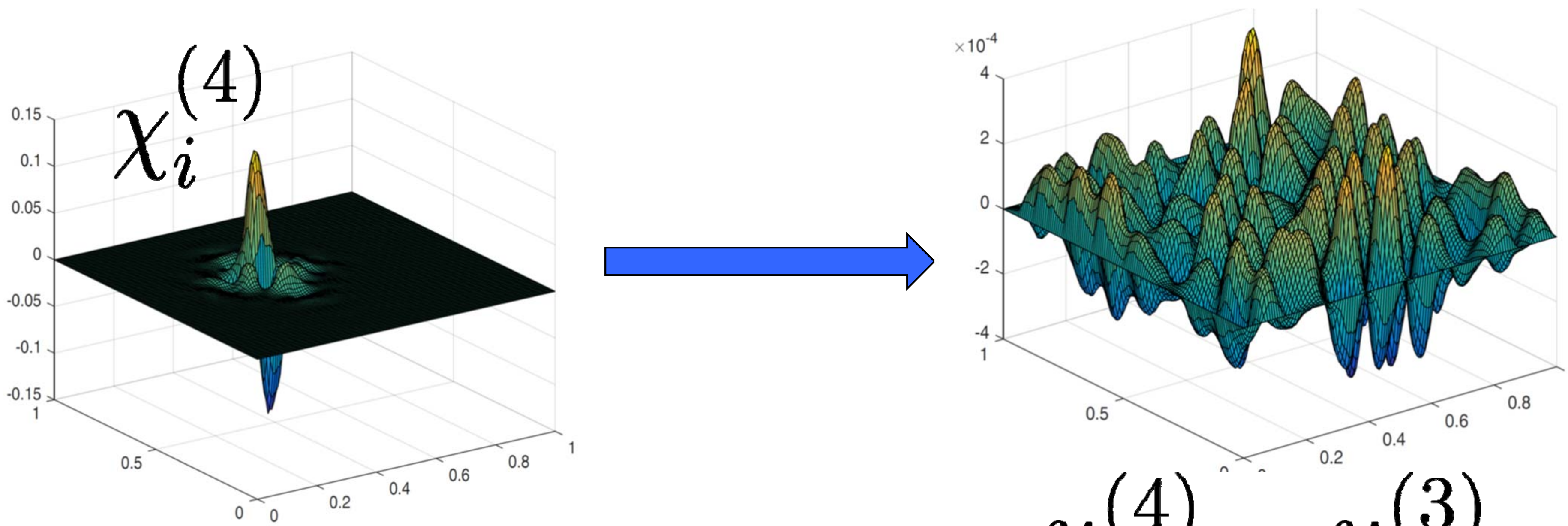
$$A^{(4)} = R^{(4,5)} A^{(5)} (R^{(4,5)})^T$$



$$\chi_i^{(4)} := \sum_{j \in \mathcal{I}^{(4)}} W_{i,j}^{(4)} \psi_j^{(4)}$$



$\chi_i^{(4)}$



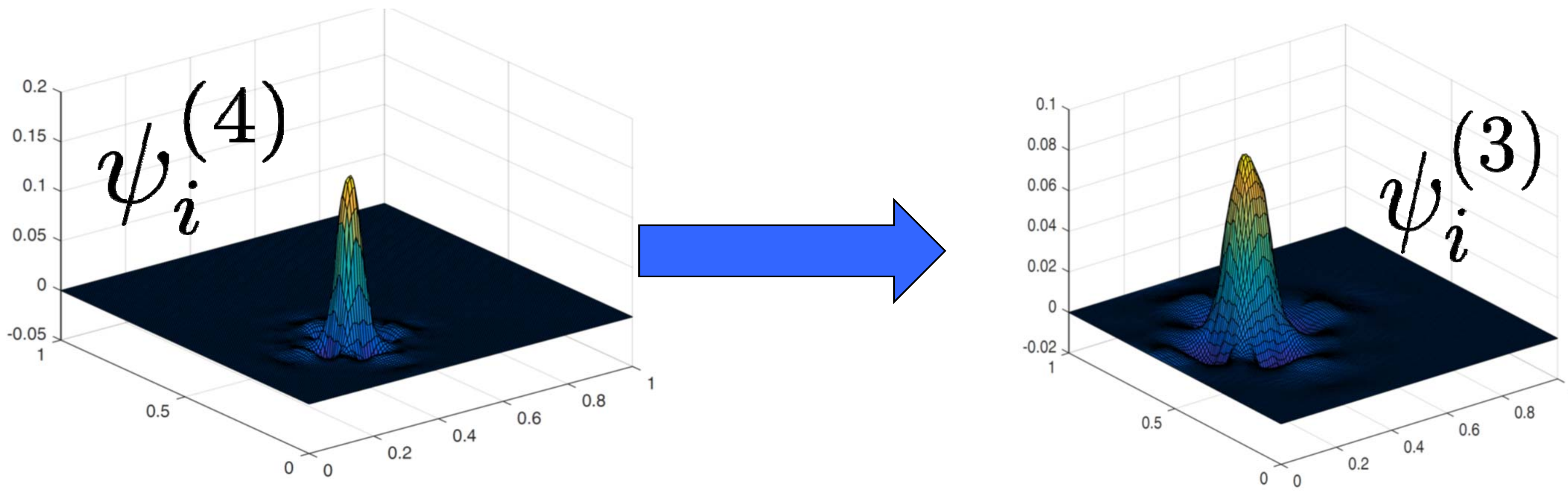
$u^{(4)} - u^{(3)}$

$$u^{(4)} - u^{(3)} = \sum_i w_i^{(4)} \chi_i^{(4)}$$

$$B^{(4)} w^{(4)} = W^{(4),T} g^{(4)}$$

$$g^{(4)} = R^{(4,5)} g^{(5)}$$

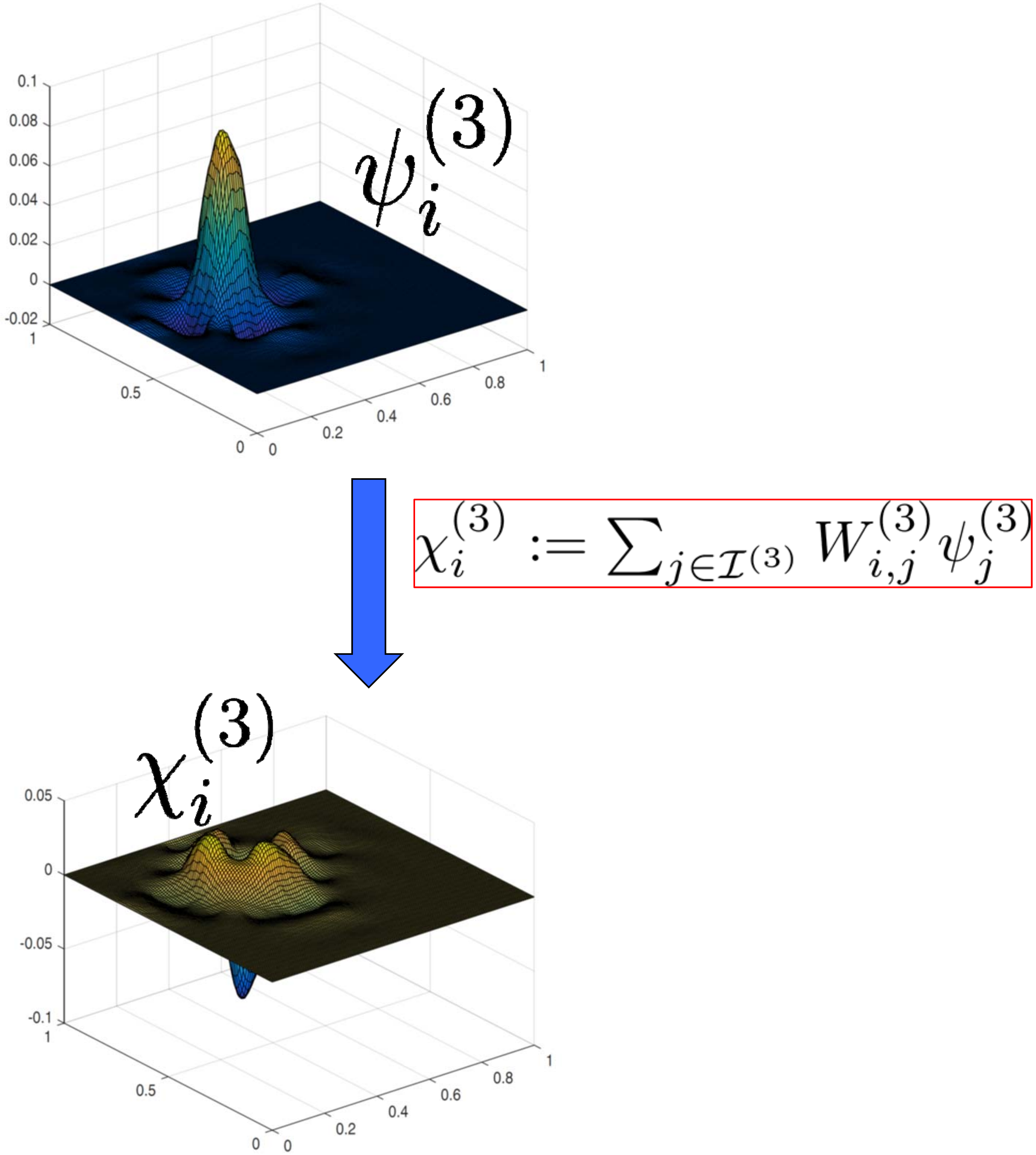
$$B^{(4)} = W^{(4)} A^{(4)} W^{(4),T}$$



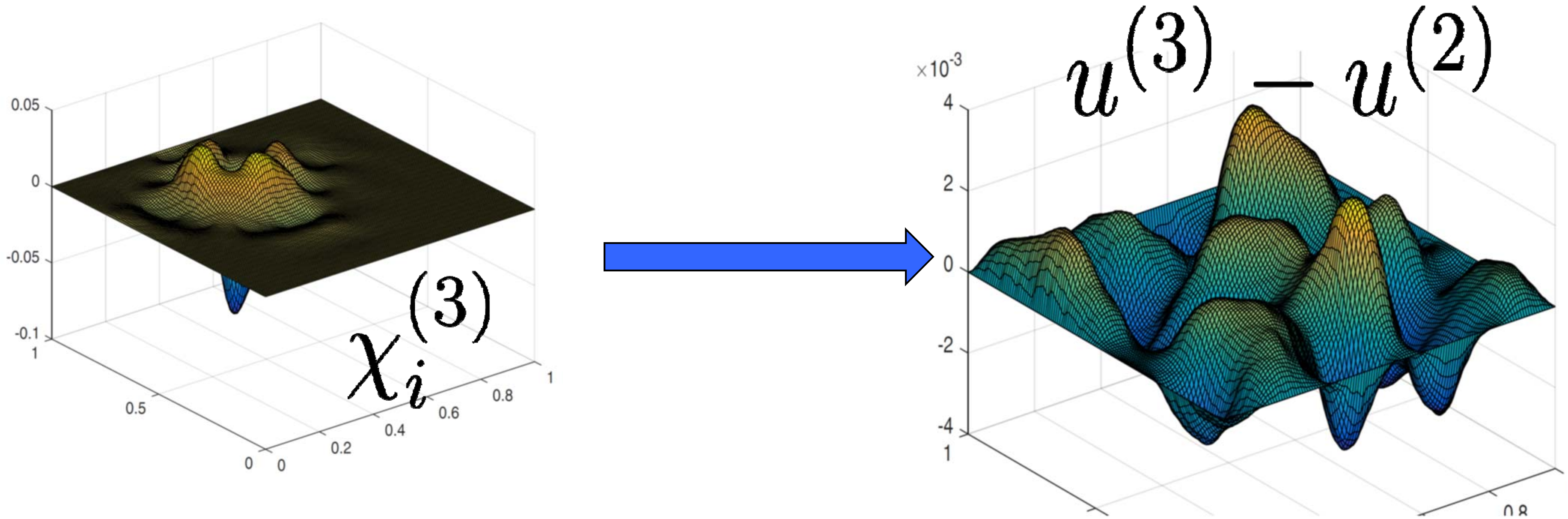
$$\psi_i^{(3)} = R_{i,j}^{(3,4)} \psi_j^{(4)}$$

$$R^{(3,4)} = \pi^{(3,4)}(I^{(4)} - A^{(4)} W^{(4),T} B^{(4),-1} W^{(4)})$$

$$A^{(3)} = R^{(3,4)} A^{(4)} (R^{(3,4)})^T$$



$\psi_i^{(3)}$
0.1
0.08
0.06
0.04
0.02
0
-0.02
1
0.5
0
0
0.2
0.4
0.6
0.8
1
$\chi_i^{(3)} := \sum_{j \in \mathcal{I}^{(3)}} W_{i,j}^{(3)} \psi_j^{(3)}$
$\chi_i^{(3)}$
0.05
0
-0.05
-0.1
1
0.5
0
0
0.2
0.4
0.6
0.8
1

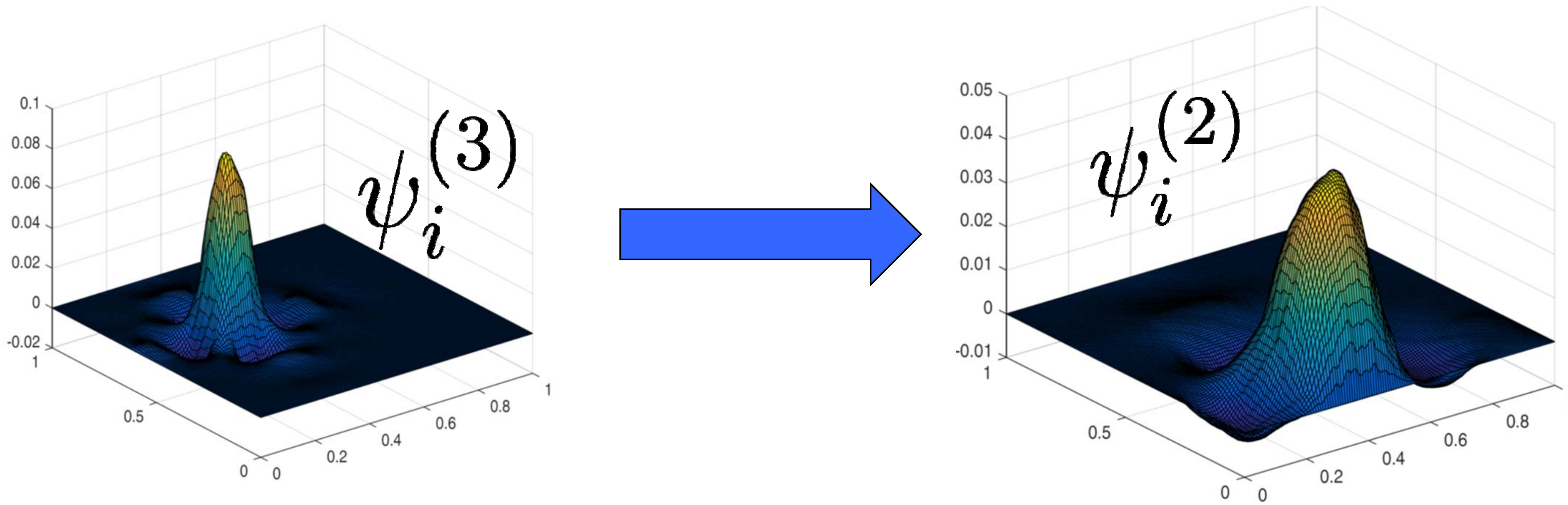


$$u^{(3)} - u^{(2)} = \sum_i w_i^{(3)} \chi_i^{(3)}$$

$$B^{(3)} w^{(3)} = W^{(3),T} g^{(3)}$$

$$g^{(3)} = R^{(3,4)} g^{(4)}$$

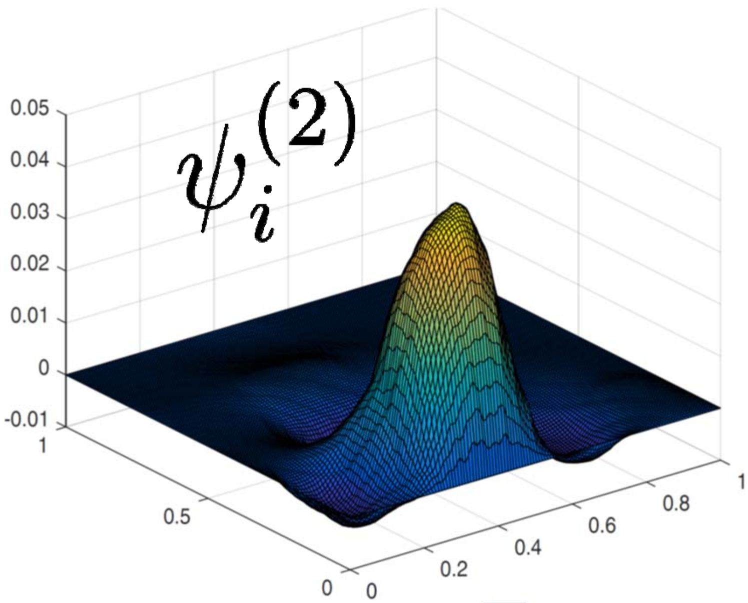
$$B^{(3)} = W^{(3)} A^{(3)} W^{(3),T}$$



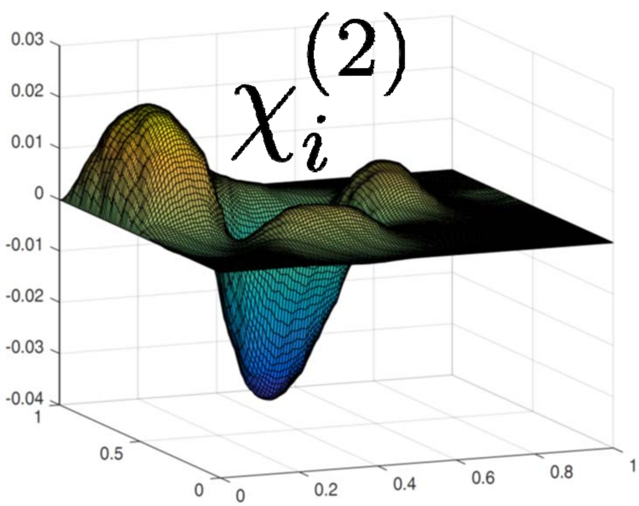
$$\psi_i^{(2)} = R_{i,j}^{(2,3)} \psi_j^{(3)}$$

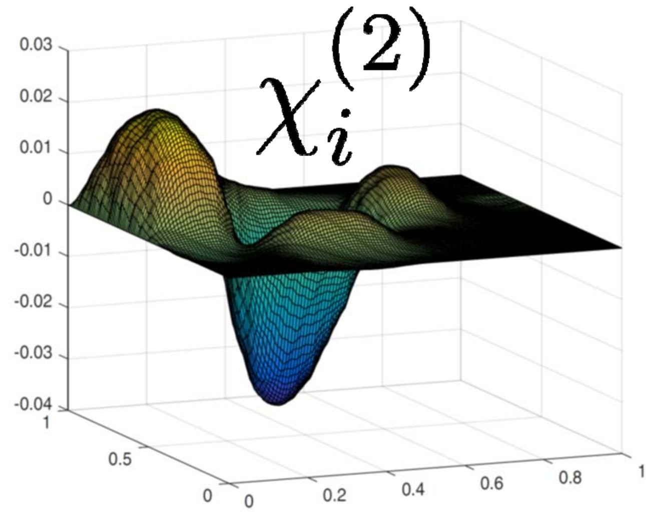
$$R^{(2,3)} = \pi^{(2,3)}(I^{(3)} - A^{(3)} W^{(3),T} B^{(3),-1} W^{(3)})$$

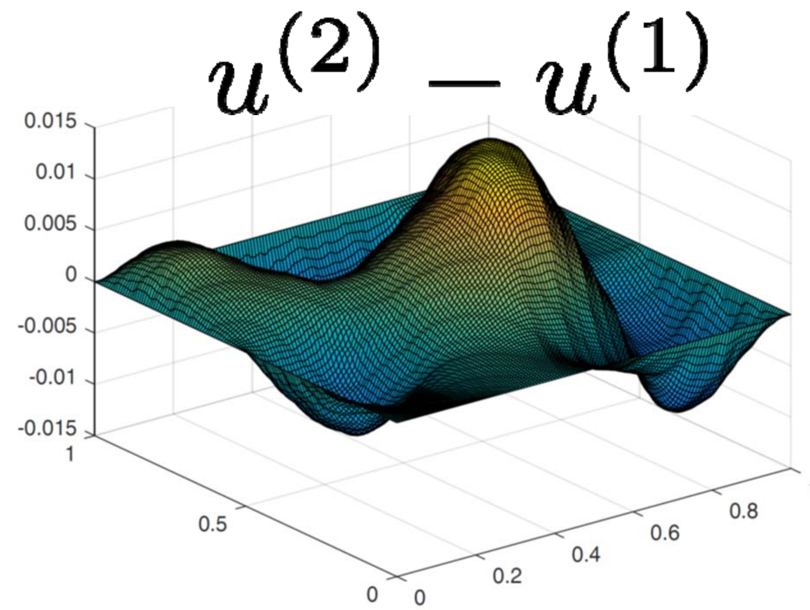
$$A^{(2)} = R^{(2,3)} A^{(3)} (R^{(2,3)})^T$$



$$\chi_i^{(2)} := \sum_{j\in\mathcal{I}^{(2)}} W_{i,j}^{(2)}\psi_j^{(2)}$$





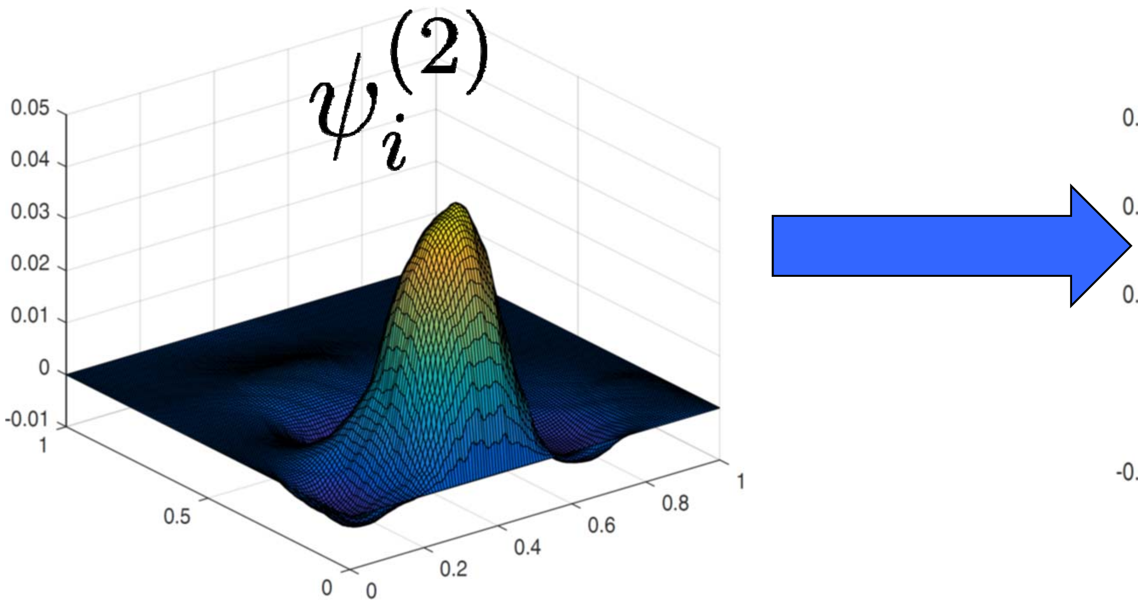


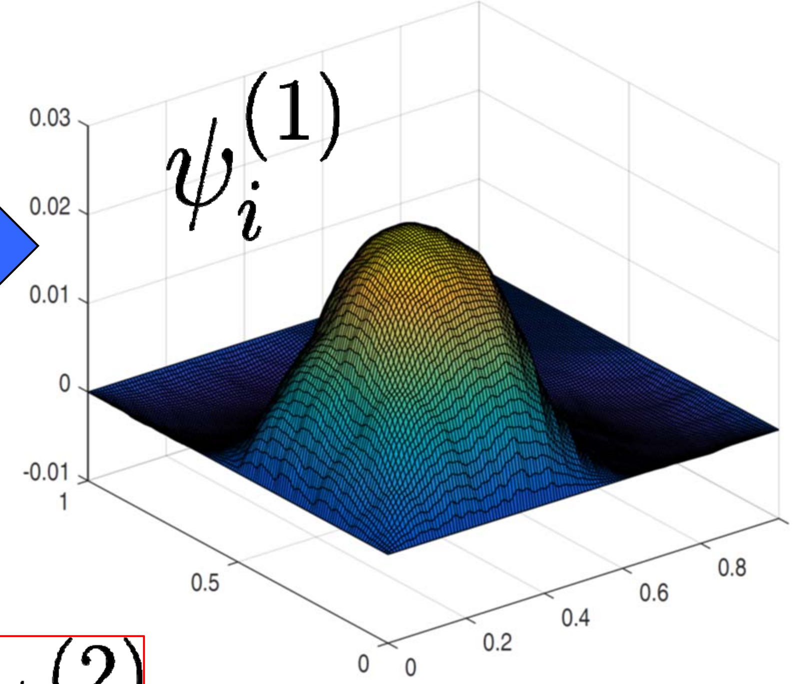
$$u^{(2)} - u^{(1)} = \sum_i w_i^{(2)} \chi_i^{(2)}$$

$$B^{(2)} w^{(2)} = W^{(2),T} g^{(2)}$$

$$g^{(2)} = R^{(2,3)} g^{(3)}$$

$$B^{(2)} = W^{(2)} A^{(2)} W^{(2),T}$$

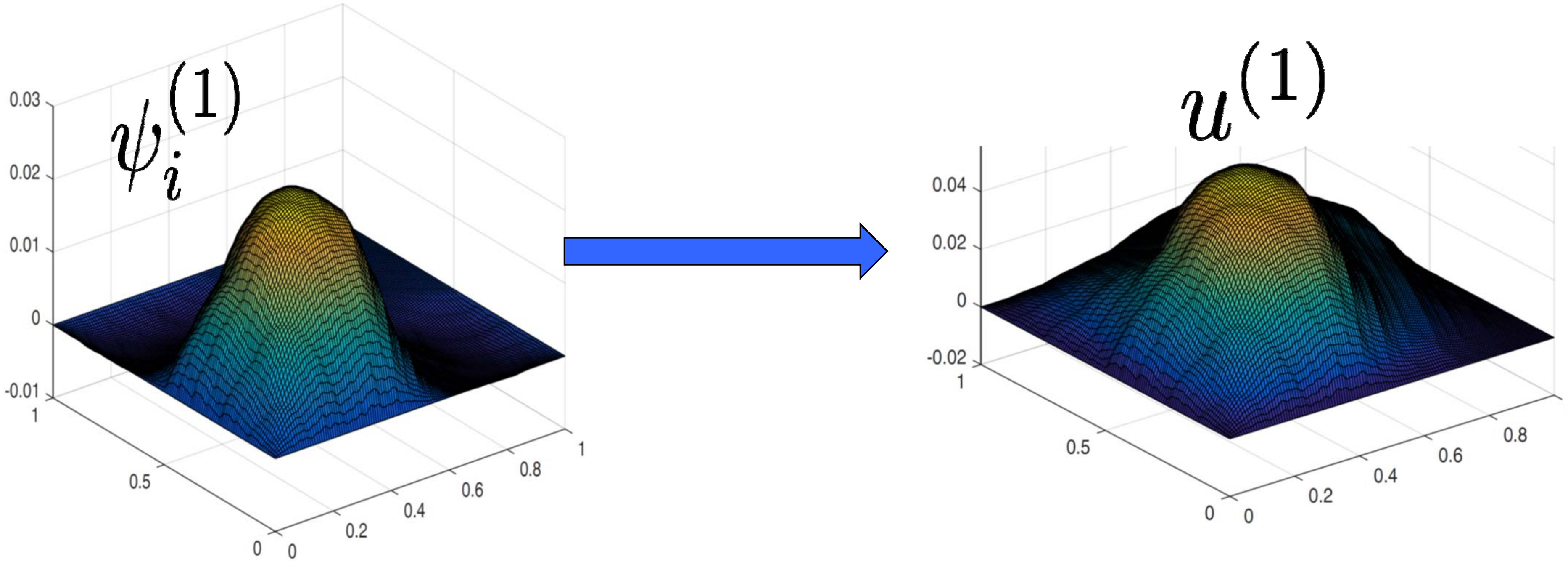




$$\psi_i^{(1)} = R_{i,j}^{(1,2)} \psi_j^{(2)}$$

$$R^{(1,2)} = \pi^{(1,2)} (I^{(2)} - A^{(2)} W^{(2),T} B^{(2),-1} W^{(2)})$$

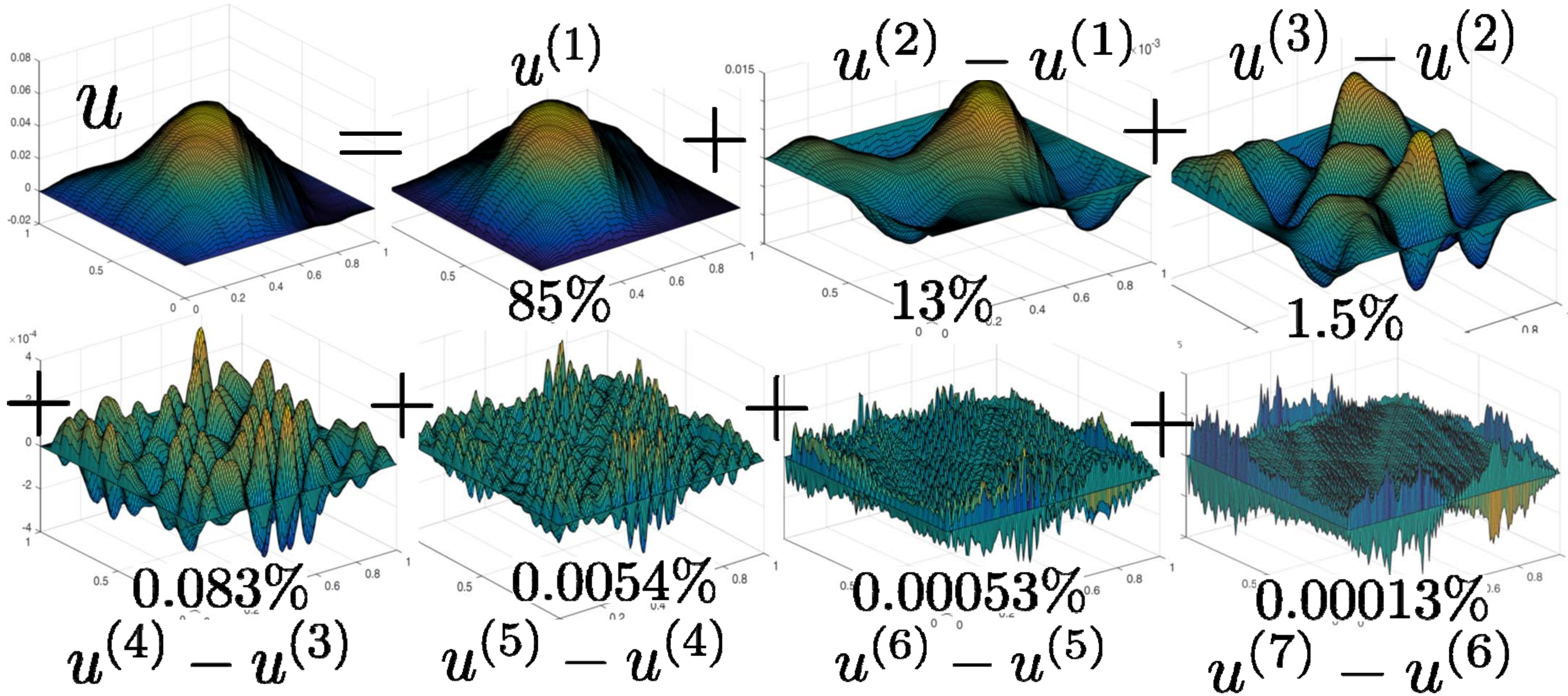
$$A^{(1)} = R^{(1,2)} A^{(2)} (R^{(1,2)})^T$$



$$u^{(1)} = \sum_i v_i^{(1)} \psi_i^{(1)}$$

$$A^{(1)} v^{(1)} = g^{(1)}$$

$$g^{(1)} = R^{(1,2)} g^{(2)}$$



$$\begin{cases} -\operatorname{div}(a\nabla u) = g, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

Application to the scalar wave PDE

[Owhadi-Zhang 2016, Gamblets for opening the complexity-bottleneck of implicit schemes for hyperbolic and parabolic PDEs with rough coefficients, arXiv:1606.07686]

$$\mu(x)\partial_t^2 u - \operatorname{div}(a\nabla u) = g(x,t)$$

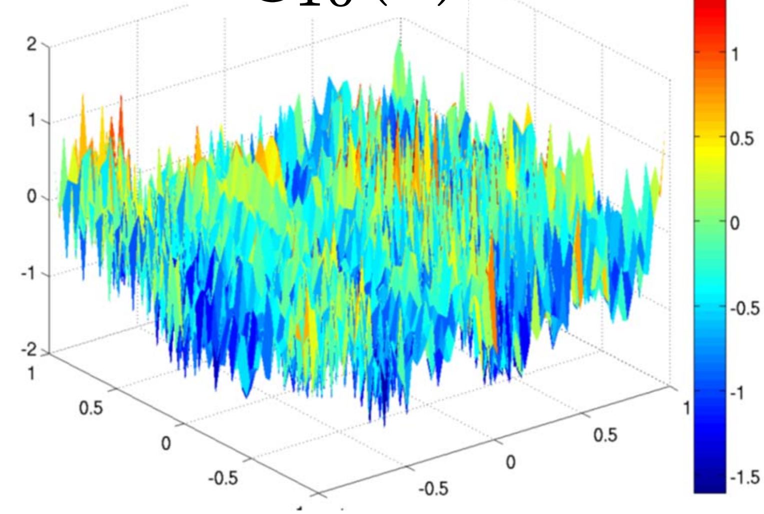
$\Omega \subset \mathbb{R}^d$

$\partial\Omega$ is piec. Lip.

a unif. ell. $a_{i,j} \in L^\infty(\Omega)$

$\mu \in L^\infty(\Omega)$ $\inf \mu > 0$

$\log_{10}(a)$



Discretization in space

$$\mu(x)\partial_t^2 u - \mathrm{div}(a\nabla u) = g(x,t)$$

Space discretization with fine scale elements $\varphi_i(x)$

$$u(x,t) = \sum_i q_i(t)\varphi_i(x)$$

$$M\ddot{q} = -Kq + f$$

$$\begin{cases} \dot{q} & = M^{-1}p \\ \dot{p} & = -Kq + f \end{cases}$$

$$M_{i,j} = \int_\Omega \varphi_i(x)\varphi_j(x)\mu(x) \qquad f_i = \int_\Omega \varphi_i(x) g(x)\,dx$$

$$K_{i,j} = \int_\Omega (\nabla\varphi_i)^T a \nabla\varphi_j$$

Time discretization with implicit midpoint

$$\begin{cases} q_{n+1} & = q_n + \Delta t M^{-1} \frac{p_n + p_{n+1}}{2} \\ p_{n+1} & = p_n - \Delta t K \frac{q_n + q_{n+1}}{2} + \Delta t \frac{f_n + f_{n+1}}{2} \end{cases}$$

2nd order accurate, implicit, A-stable

Preserves quadratic invariants exactly

Total energy $\boxed{E = \frac{1}{2} p^T M^{-1} p + \frac{1}{2} q^T K q}$

Energy in each vibration mode/eigenfunction

Time discretization with implicit midpoint

$$\begin{cases} (M + \frac{(\Delta t)^2}{4} K) q_{n+1} & = (M - \frac{(\Delta t)^2}{4} K) q_n + \Delta t p_n + \Delta t^2 \frac{f_n + f_{n+1}}{4} \\ p_{n+1} & = p_n - \Delta t K \frac{q_n + q_{n+1}}{2} + \Delta t \frac{f_n + f_{n+1}}{2} \end{cases}$$

Solving $(M + \frac{\Delta t^2}{4} K) q = b$

⇕

Solving $\frac{4}{\zeta^2} \mu u - \operatorname{div}(a \nabla u) = g$ with $\zeta = \Delta t$

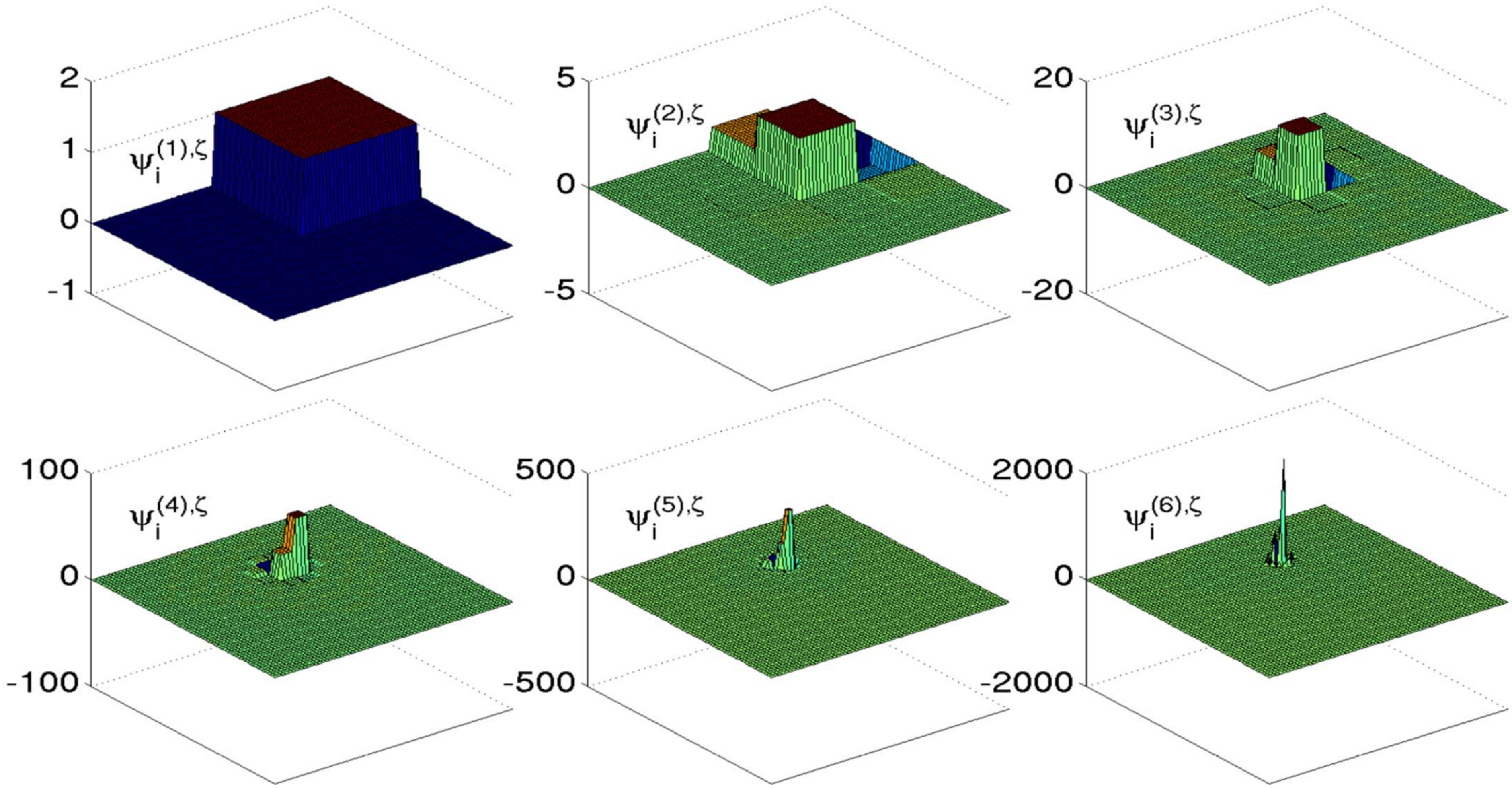
Klein-Gordon PDE

⇕

Change of basis: Localized ζ-Gamblets

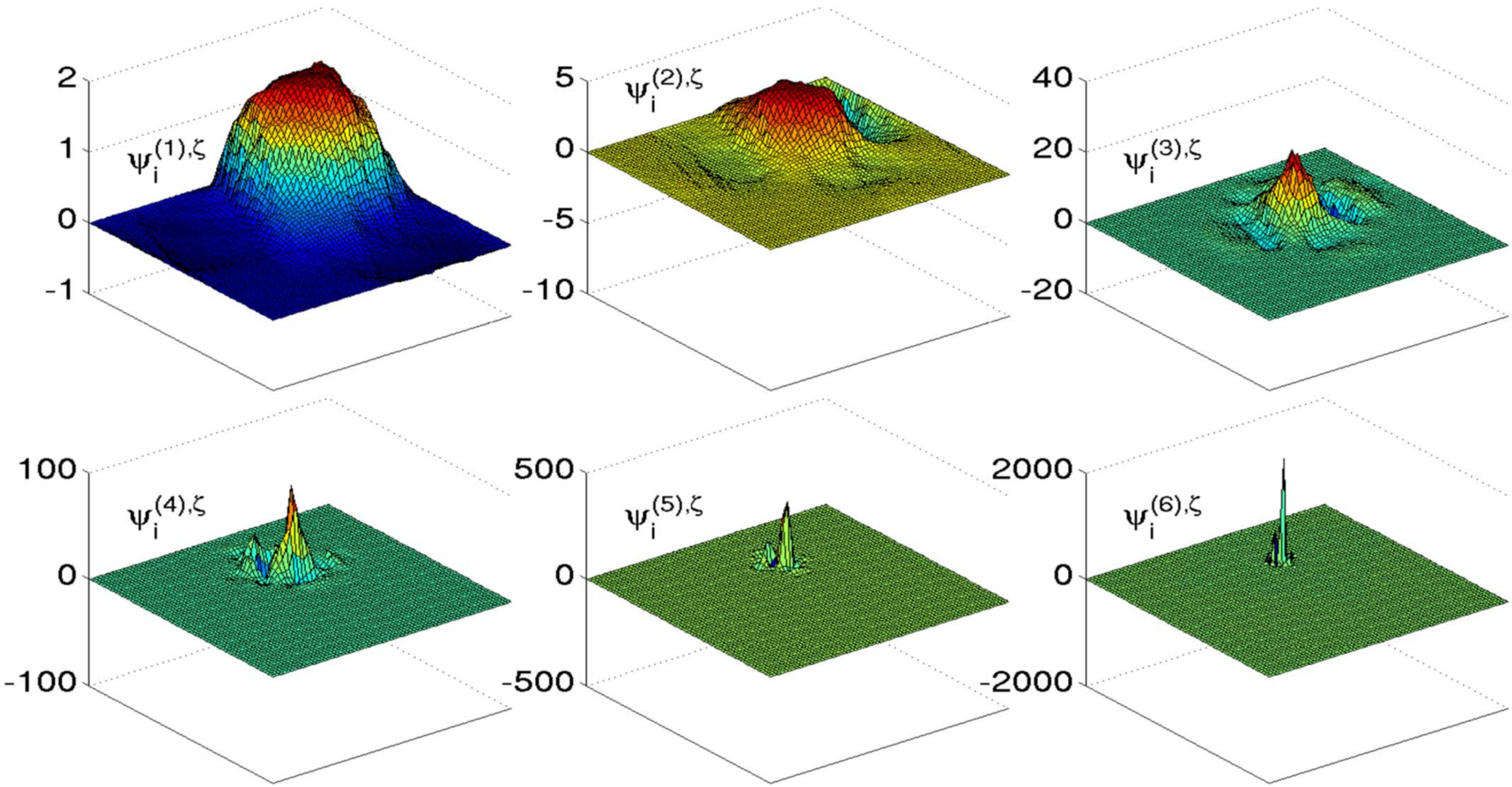
$$\min_\psi \frac{4}{\zeta^2} \int_\Omega \psi^2 \mu + \int_\Omega (\nabla \psi)^T a \nabla \psi \text{ s.t. } \int_\Omega \psi \phi_j = \delta_{i,j}$$

Gamblets



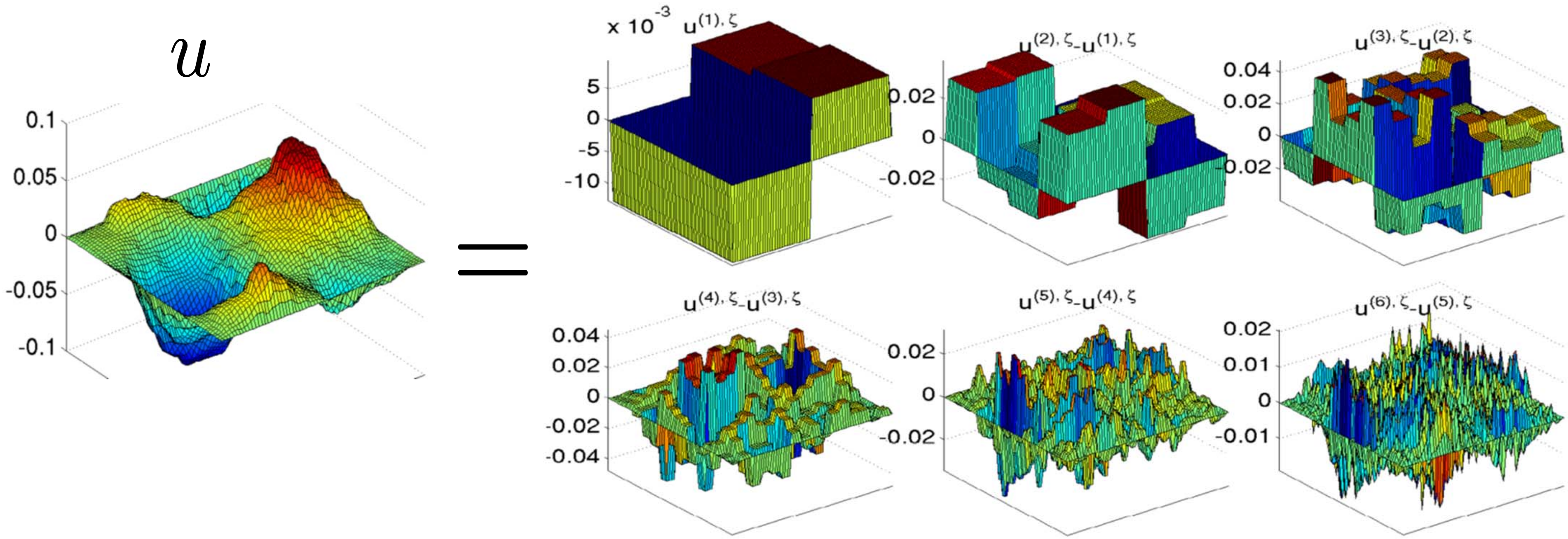
$$\zeta = \Delta t = 10^{-6}$$

Gamblets



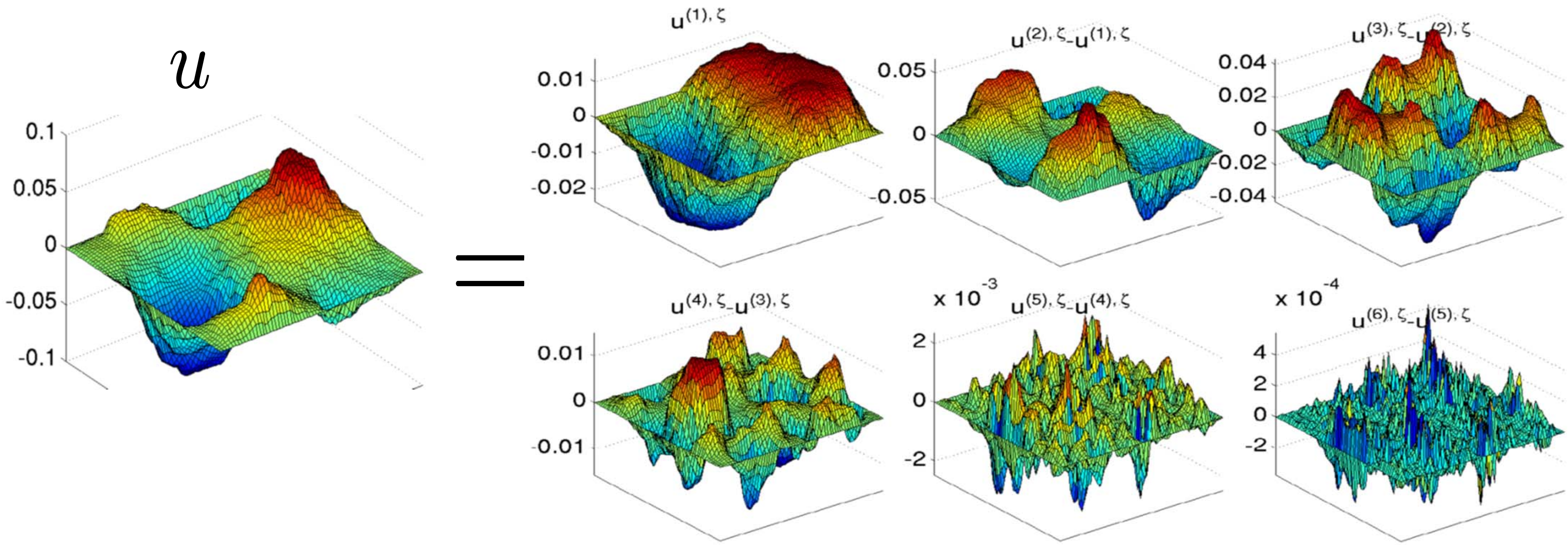
$$\zeta = \Delta t = 1$$

Multiresolution decomposition of the solution



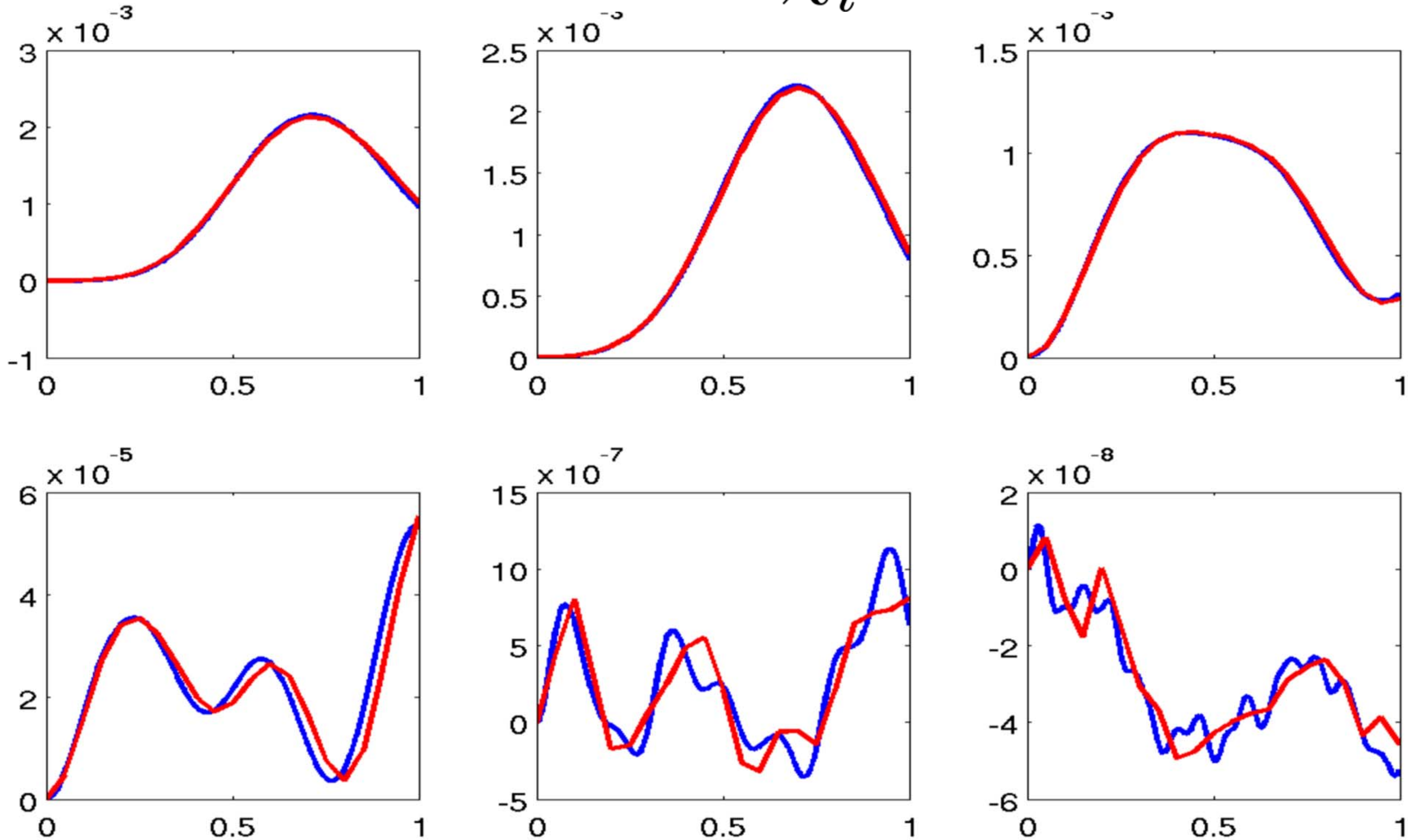
$$\zeta = \Delta t = 10^{-6}$$

Multiresolution decomposition of the solution



$$\zeta = \Delta t = 1$$

Evolution of the $\chi_i^{(k)}$ coefficient



Blue: Reference solution

Red: Implicit midpoint with localized gamblets (3 layers)

Time discretization with 2 stages Gauss-Legendre

Butcher tableau for GL2

c	A	
$\frac{1}{2}-\frac{\sqrt{3}}{6}$	$\frac{1}{4}$	$\frac{1}{4}-\frac{\sqrt{3}}{6}$
$\frac{1}{2}+\frac{\sqrt{3}}{6}$	$\frac{1}{4}+\frac{\sqrt{3}}{6}$	$\frac{1}{4}$
b	$\frac{1}{2}$	$\frac{1}{2}$

$$k_n^1 = f_n^1 + \begin{pmatrix} \mathbf{0} & M^{-1} \\ -K & \mathbf{0} \end{pmatrix} (y_n + \Delta t A_{11} k_n^1 + \Delta t A_{12} k_n^2)$$

$$k_n^2 = f_n^2 + \begin{pmatrix} \mathbf{0} & M^{-1} \\ -K & \mathbf{0} \end{pmatrix} (y_n + \Delta t A_{21} k_n^1 + \Delta t A_{22} k_n^2)$$

$$y_{n+1} = y_n + \Delta t(b_1 k_n^1 + b_2 k_n^2)$$

$$y_n = (q_n; p_n) \qquad f_n^1 = (\mathbf{0}; f(t_n + c_1 h)) \quad f_n^2 = (\mathbf{0}; f(t_n + c_2 h))$$

4th order accurate, unconditionally stable, symplectic, symmetric (time-reversible) and preserves quadratic invariants exactly

Time discretization with 2 stages Gauss-Legendre

One time step of 2 stages Gauss-Legendre

$\Updownarrow$

Solving $(M + \Delta t^2 \lambda_i^2 K) q_i = b_i$ with $i = 1, 2$

λ_i: complex eigenvalues of $A = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{3}}{6} \\ \frac{1}{4} + \frac{\sqrt{3}}{6} & \frac{1}{4} \end{pmatrix}$

$\Updownarrow$

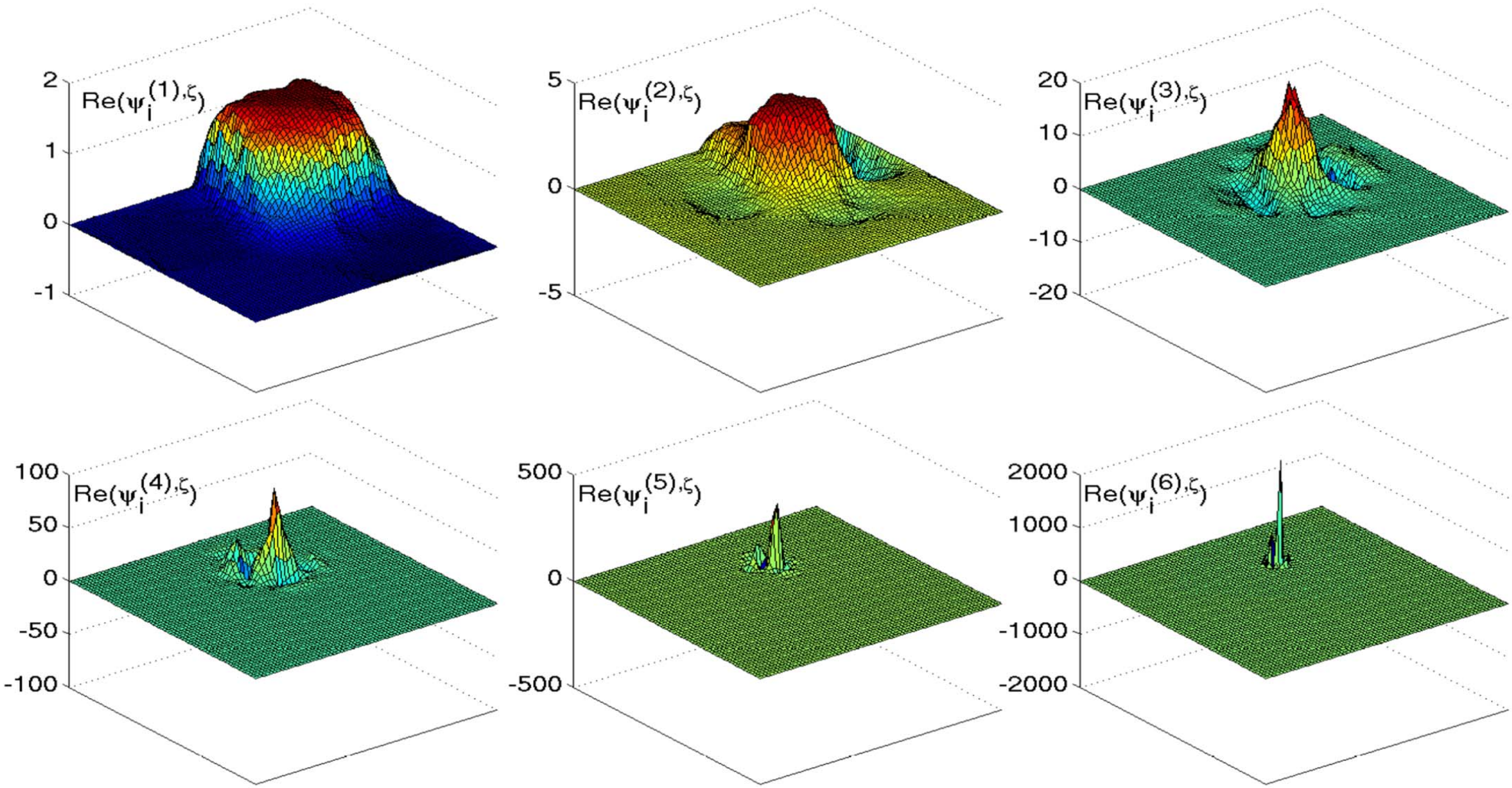
Solving $\frac{4\mu}{\zeta_i^2} u - \operatorname{div}(a \nabla u_i) = g_i$ with $\zeta = 2\lambda_i \Delta t$

$\zeta \in \mathbb{C}/\{0\}$

$\Updownarrow$

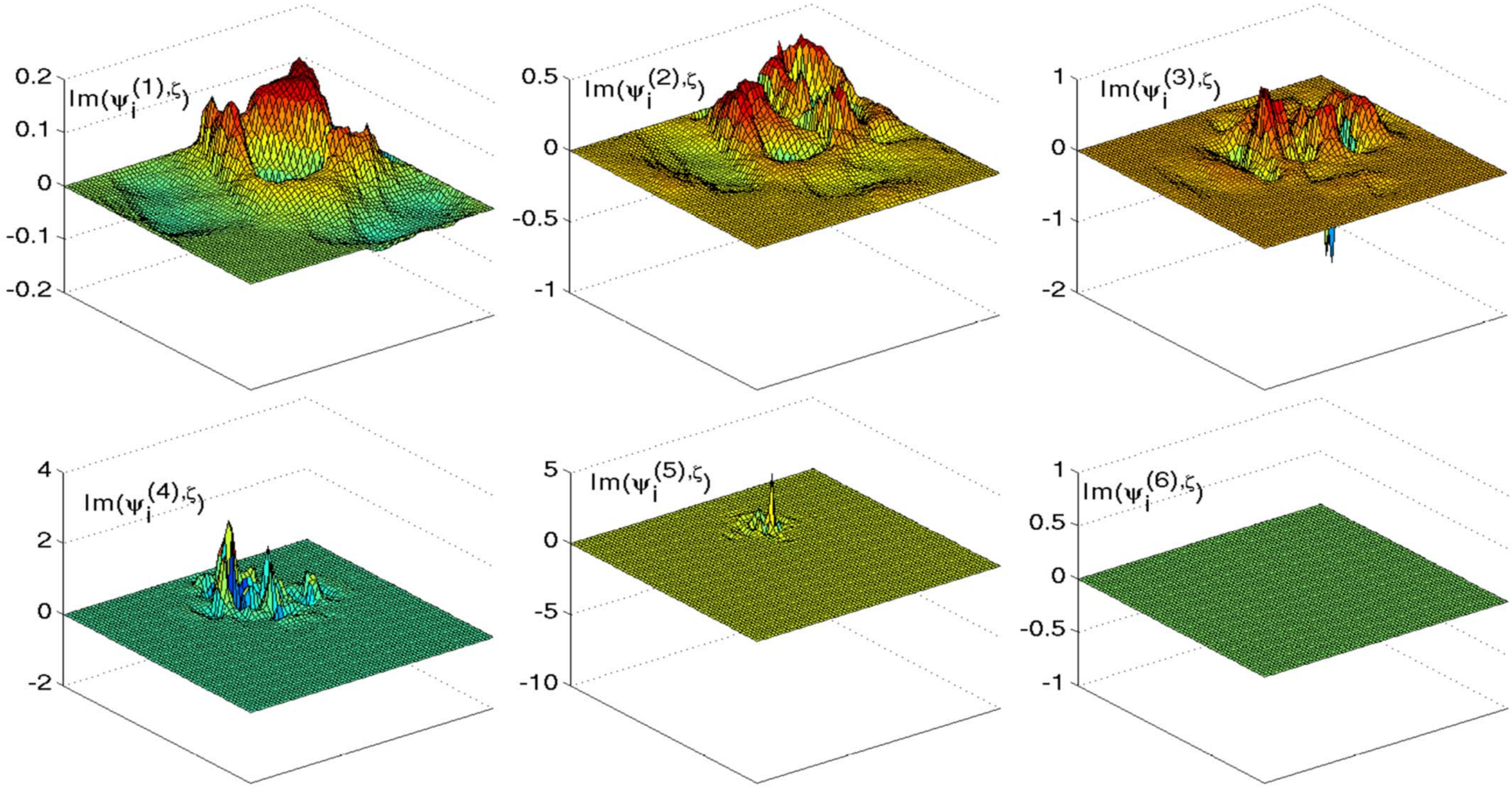
Change of basis: Localized ζ-Gamblets

Complex Gamblets (real part)



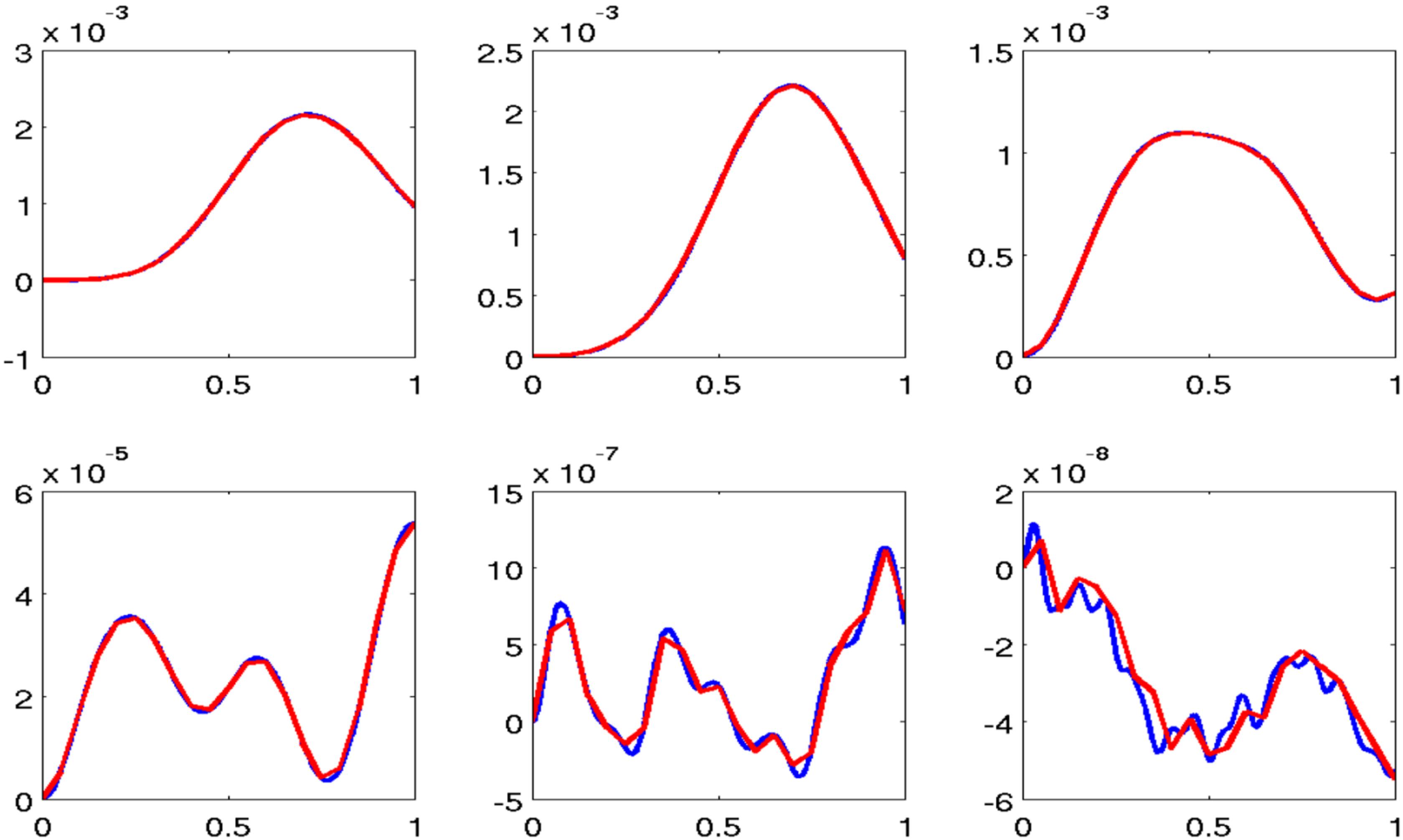
Real part of $\psi_i^{(k)}$ asssociated with the eigenvalue $0.1626 + 0.1849\,i$ and $\Delta t = 0.1$

Complex Gamblets (imaginary part)



Imaginary part of $\psi_i^{(k)}$ asssociated with the eigenvalue $0.1626 + 0.1849\, i$ and $\Delta t = 0.1$

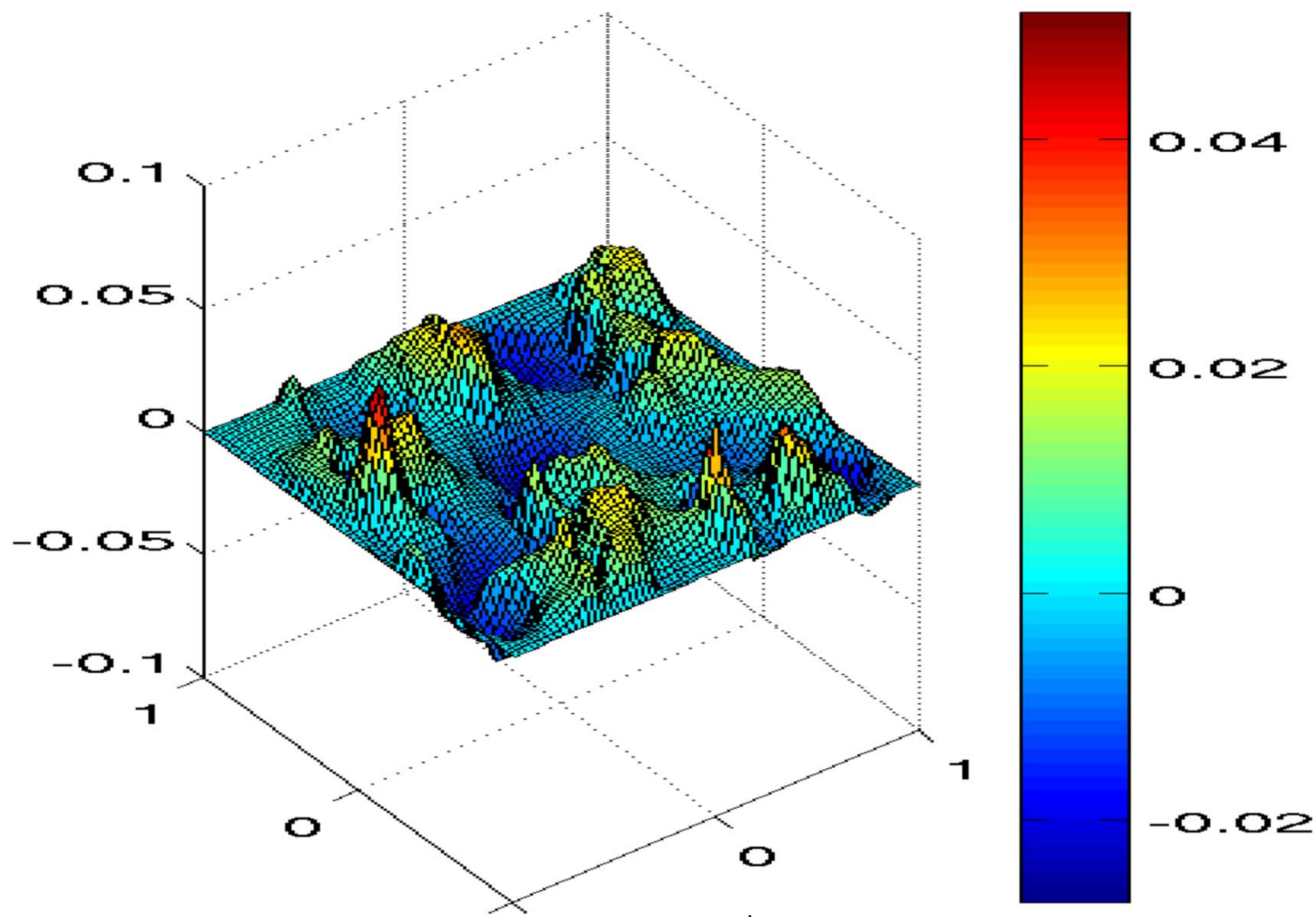
Evolution of the $\chi_i^{(k)}$ coefficient



Blue: Reference solution

Red: 2 stages Gauss-Legendre with localized gamblets (3 layers)

u at $T = 1$

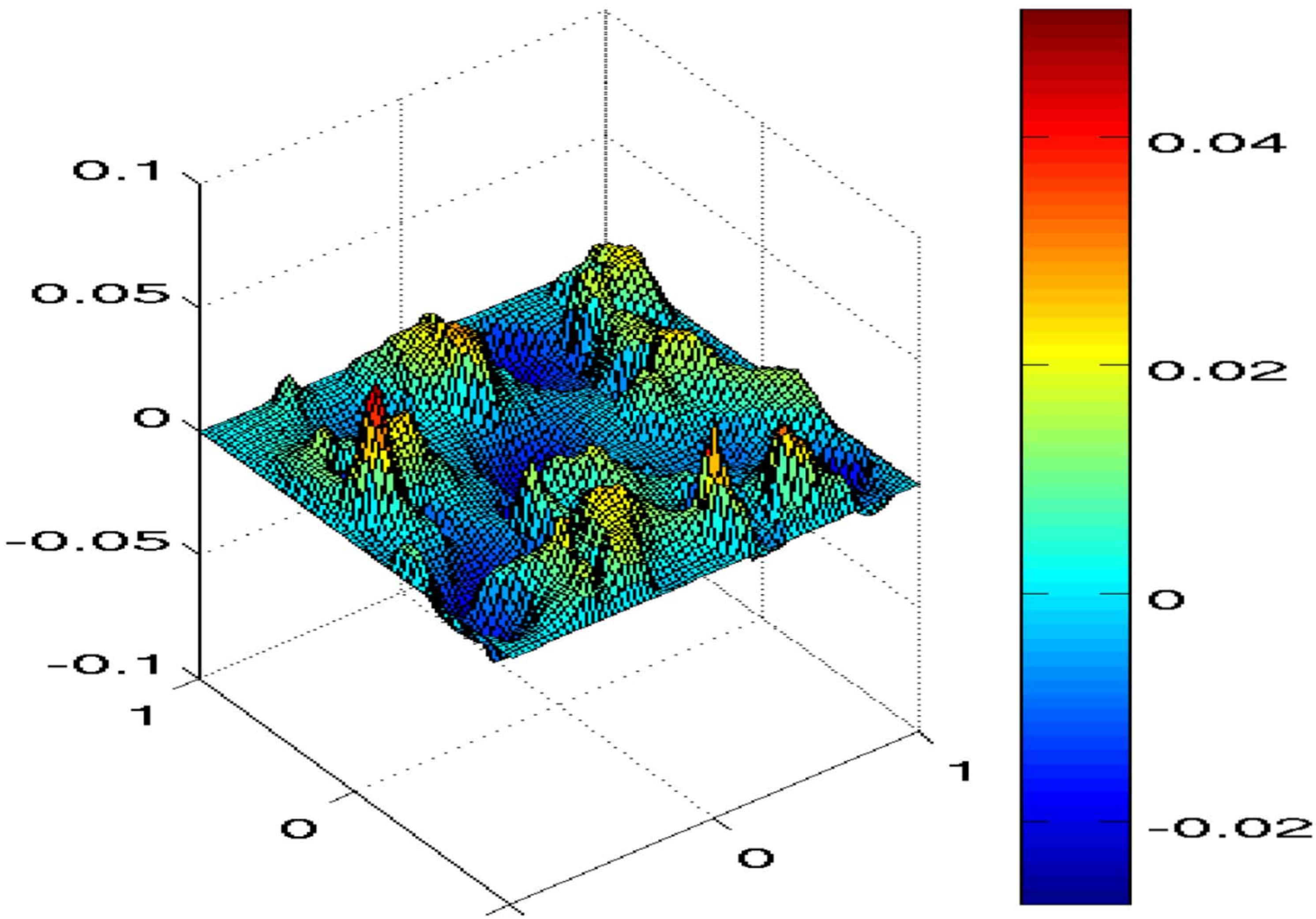


ode15s with $dt = 1/1280$

$g(x,t) = \sin(2\pi(t + x_1))\cos(2\pi(t + x_2))$

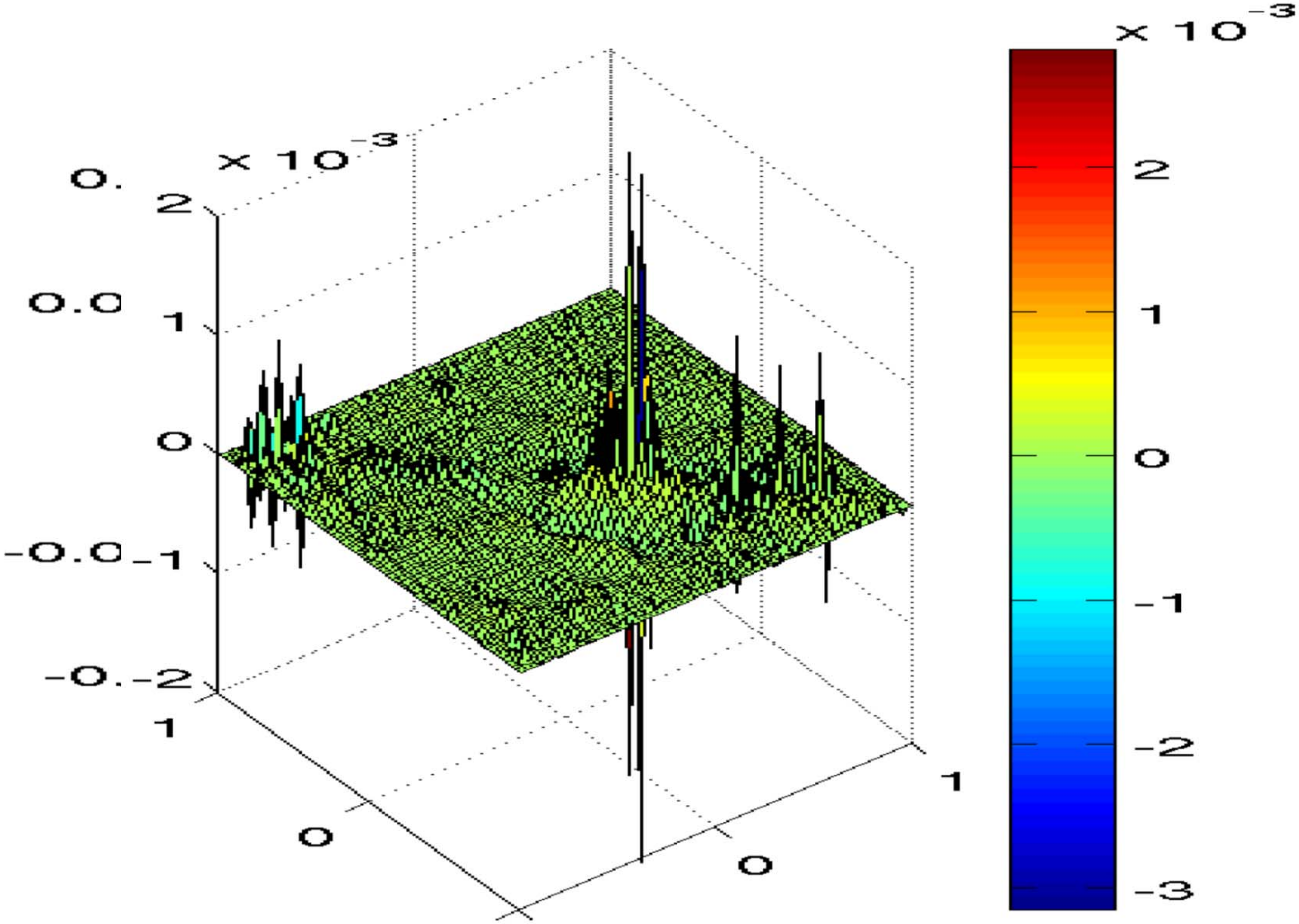
$u(x,0) = 0$ and $u_t(x,0) = \sin(2\pi x_1)\cos(2\pi x_2)$

$\bar{u}$ at $T = 1$



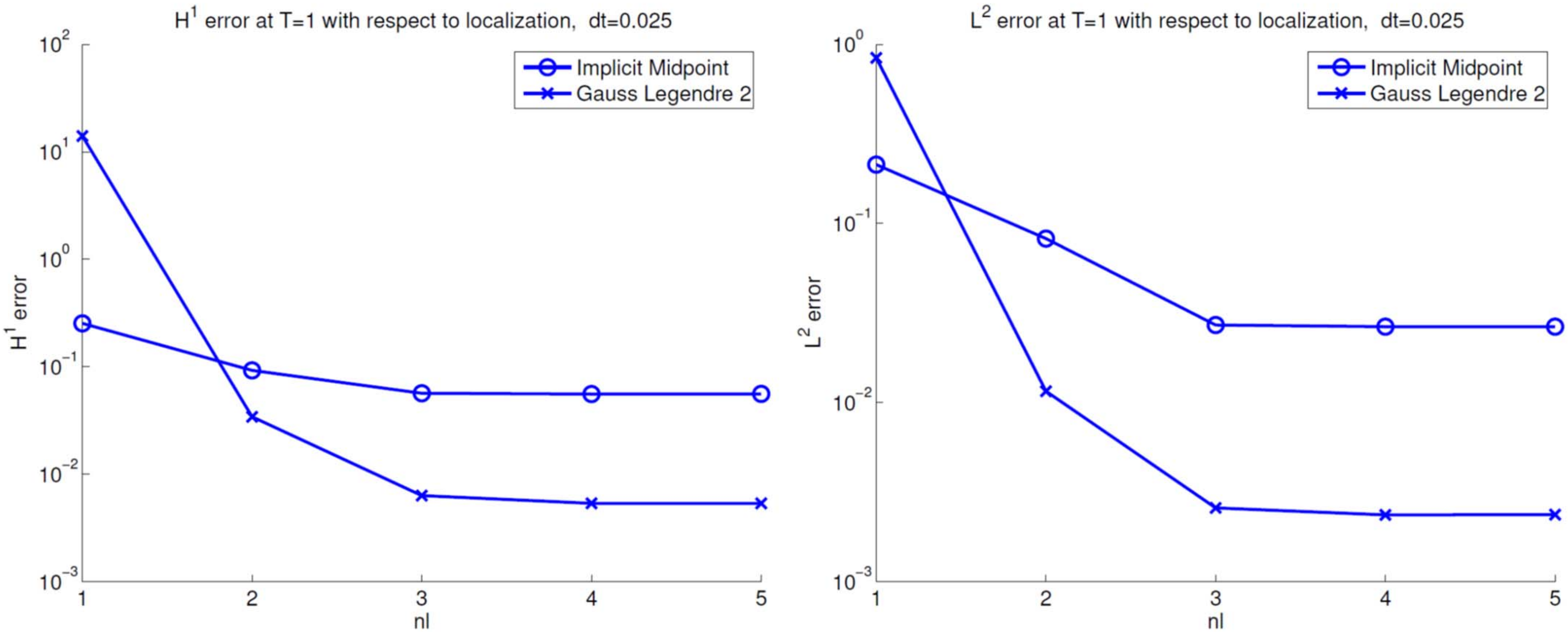
2 stages Gauss-Legendre with $\Delta t = 0.1$

$$u - \bar{u} \text{ at } T = 1$$



$$\frac{\Delta t}{dt} = 128$$

Error vs localization ($T = 1$, $dt = 0.025$)



H^1 error $\qquad$ L^2 error

$N(nl)^d$ complexity

Parabolic PDE

$$\mu(x)\partial_t u - \operatorname{div}(a\nabla u) = g(x,t)$$

Space discretization with fine scale elements $\varphi_i(x)$

$$u(x,t) = \sum_i q_i(t)\varphi_i(x)$$

$$M\dot{q} = -Kq + f$$

$$M_{i,j} = \int_\Omega \varphi_i(x)\varphi_j(x)\mu(x) \qquad f_i = \int_\Omega \varphi_i(x)g(x)\,dx$$

$$K_{i,j} = \int_\Omega (\nabla\varphi_i)^T a \nabla\varphi_j$$

$h \sim N^{-1/d}$: Resolution of the fine mesh

Time discretization with explicit scheme requires $\Delta t \sim h^2$

$$M\dot{q} = -Kq + f$$

Time discretization with TR-BDF2

$$\begin{cases} (M + \frac{\gamma \Delta t}{2} K) q_{n+\gamma} = (M - \frac{\gamma \Delta t}{2} K) q_n + \Delta t \frac{f_n + f_{n+\gamma}}{2} \\ (M + \frac{1-\gamma}{2-\gamma} \Delta t K) q_{n+1} = \frac{1}{\gamma(2-\gamma)} M q_{n+\gamma} - \frac{(1-\gamma)^2}{\gamma(2-\gamma)} M q_n + \frac{1-\gamma}{2-\gamma} \Delta t f_{n+1} \end{cases}$$

2nd order accurate, implicit and L-stable

$\gamma = 2 - \sqrt{2}$ → Minimizes local truncation error

→ implies $\frac{\gamma}{2} = \frac{1-\gamma}{2-\gamma}$

[R.E. Bank, Jr. Coughran, W.M., Wolfgang Fichtner, E.H. Grosse, D.J. Rose, and R.K. Smith. Transient simulation of silicon devices and circuits. Computer-Aided Design of Integrated Circuits and Systems, IEEE Transactions on, 4(4):436451, October 1985.]

$$M\dot{q} = -Kq + f$$

Time discretization with TR-BDF2

$$\begin{cases} (M + \frac{\gamma \Delta t}{2} K) q_{n+\gamma} = (M - \frac{\gamma \Delta t}{2} K) q_n + \Delta t \frac{f_n + f_{n+\gamma}}{2} \\ (M + \frac{1-\gamma}{2-\gamma} \Delta t K) q_{n+1} = \frac{1}{\gamma(2-\gamma)} M q_{n+\gamma} - \frac{(1-\gamma)^2}{\gamma(2-\gamma)} M q_n + \frac{1-\gamma}{2-\gamma} \Delta t f_{n+1} \end{cases}$$

Solving $(M + \frac{\gamma \Delta t}{2} K) q = b$

⇕

Solving $\frac{4}{\zeta^2} u - \mathrm{div}(a \nabla u) = g$ with $\zeta = \sqrt{\gamma \Delta t / 2}$

⇕

Change of basis: Localized ζ-Gamblets

$$\min_\psi \frac{4}{\zeta^2} \int_\Omega \psi^2 \mu + \int_\Omega (\nabla \psi)^T a \nabla \psi \text{ s.t. } \int_\Omega \psi \phi_j = \delta_{i,j}$$

Implicit Methods for Time Dependent Problems

For parabolic equation, need implicit method for better stability, e.g., B-stability (contractive) or L-stability (stiff accurate).

- Implicit Euler, $(M + \Delta t K)q_{n+1} = M q_n + \Delta t f_{n+1}$ ($\zeta = 2\sqrt{\Delta t}$)
- TR-BDF2 (L-stable and 2nd order),

$$(M + \frac{\gamma \Delta t}{2} K) q_{n+\gamma} = (M - \frac{\gamma \Delta t}{2} K) q_n + \Delta t \frac{f_n + f_{n+\gamma}}{2}$$

$$(M + \frac{1-\gamma}{2-\gamma} \Delta t K) q_{n+1} = \frac{1}{\gamma(2-\gamma)} M q_{n+\gamma} - \frac{(1-\gamma)^2}{\gamma(2-\gamma)} M q_n + \frac{1-\gamma}{2-\gamma} \Delta t f_{n+1}$$

- SDIRK (Singly Diagonally Implicit Runge-Kutta) and DIRK (Diagonally Implicit Runge-Kutta). L-stable, B-stable and 3rd order accurate.

λ	λ	0	0
$\frac{1}{2}(1+\lambda)$	$\frac{1}{2}(1-\lambda)$	λ	0
1	$\frac{1}{4}(-6\lambda^2 + 16\lambda - 1)$	$\frac{1}{4}(6\lambda^2 - 20\lambda + 5)$	λ
	$\frac{1}{4}(-6\lambda^2 + 16\lambda - 1)$	$\frac{1}{4}(6\lambda^2 - 20\lambda + 5)$	λ

Table 1: *

Butcher tableau for SDIRK3 where $\lambda \simeq 0.4358665215$ (identified as a root of $\frac{1}{6} - \frac{3}{2}\lambda + 3\lambda^2 - \lambda^3 = 0$) ensures L-stability.

Fully implicit Runge-Kutta methods

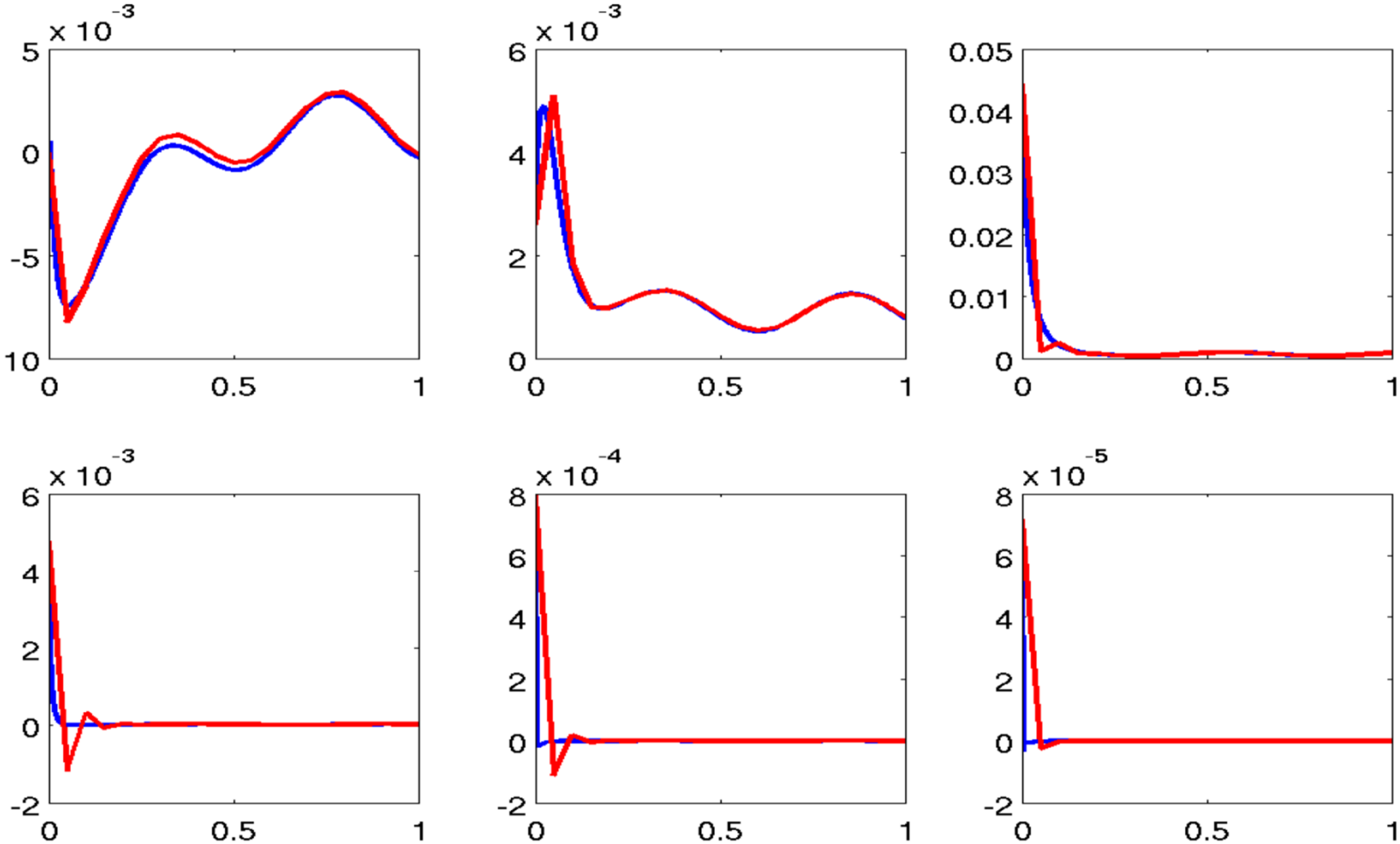
Lobatto IIIC (4th order, L-stable, B-stable, and stiffly accurate)

$$\begin{array}{c|ccc} 0 & 1/6 & -1/3 & 1/6 \\ 0.5 & 1/6 & 5/12 & -1/12 \\ 1 & 1/6 & 2/3 & 1/6 \\ \hline & 1/6 & 2/3 & 1/6 \end{array}$$

Radau IIA (5th order, A-stable)

$$\begin{array}{c|ccc} \frac{2}{5} - \frac{\sqrt{6}}{10} & \frac{11}{45} - \frac{7\sqrt{6}}{360} & \frac{37}{225} - \frac{169\sqrt{6}}{1800} & -\frac{2}{225} + \frac{\sqrt{6}}{75} \\ \frac{2}{5} + \frac{\sqrt{6}}{10} & \frac{37}{225} + \frac{169\sqrt{6}}{1800} & \frac{11}{45} + \frac{7\sqrt{6}}{360} & -\frac{2}{225} - \frac{\sqrt{6}}{75} \\ 1 & \frac{4}{9} - \frac{\sqrt{6}}{36} & \frac{4}{9} + \frac{\sqrt{6}}{36} & \frac{1}{9} \\ \hline & \frac{4}{9} - \frac{\sqrt{6}}{36} & \frac{4}{9} + \frac{\sqrt{6}}{36} & \frac{1}{9} \end{array}$$

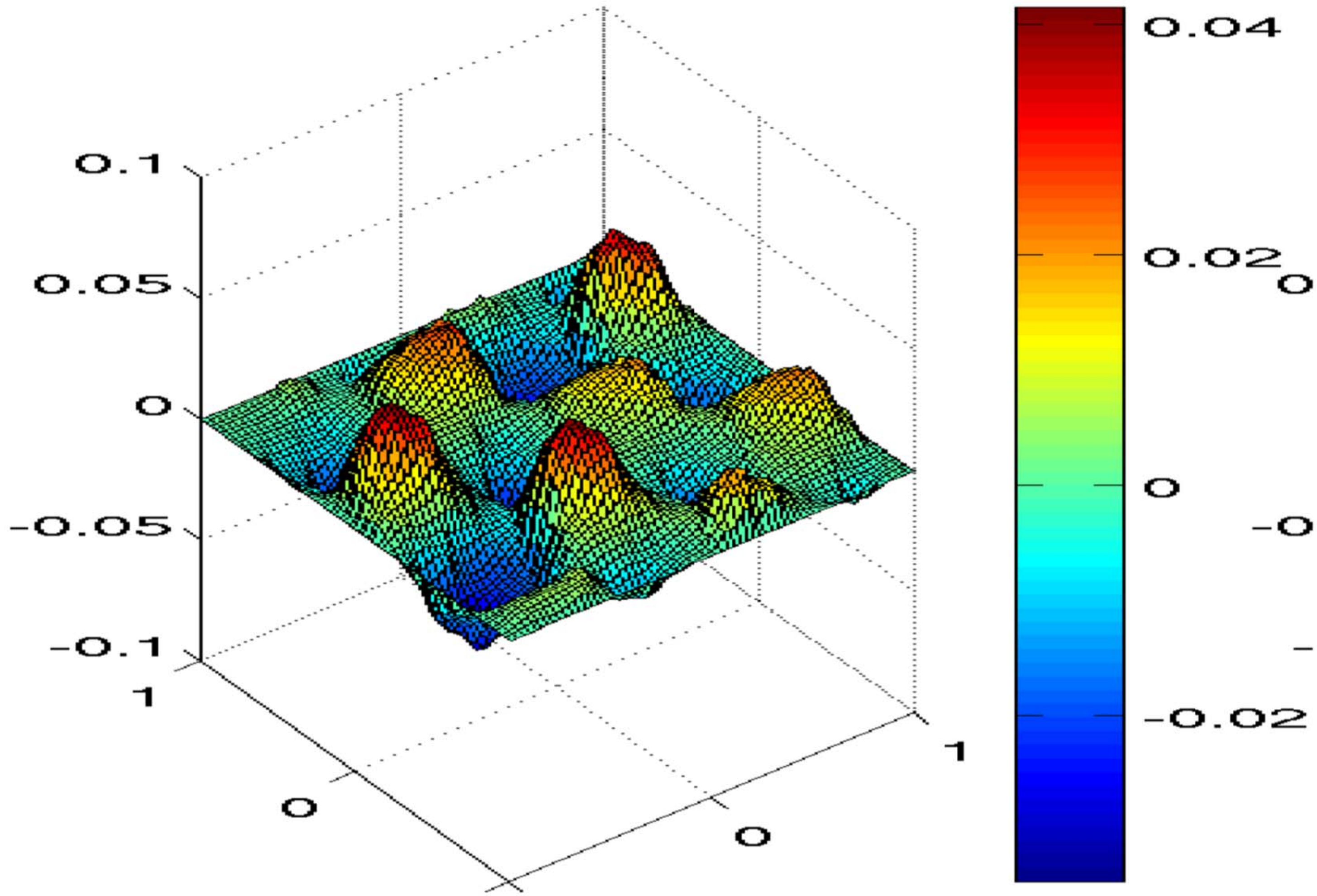
Evolution of the $\chi_1^{(k)}$ coefficient



Blue: Reference solution

Red: DIRK3 with localized gamblets (3 layers)

u at $T = 1$

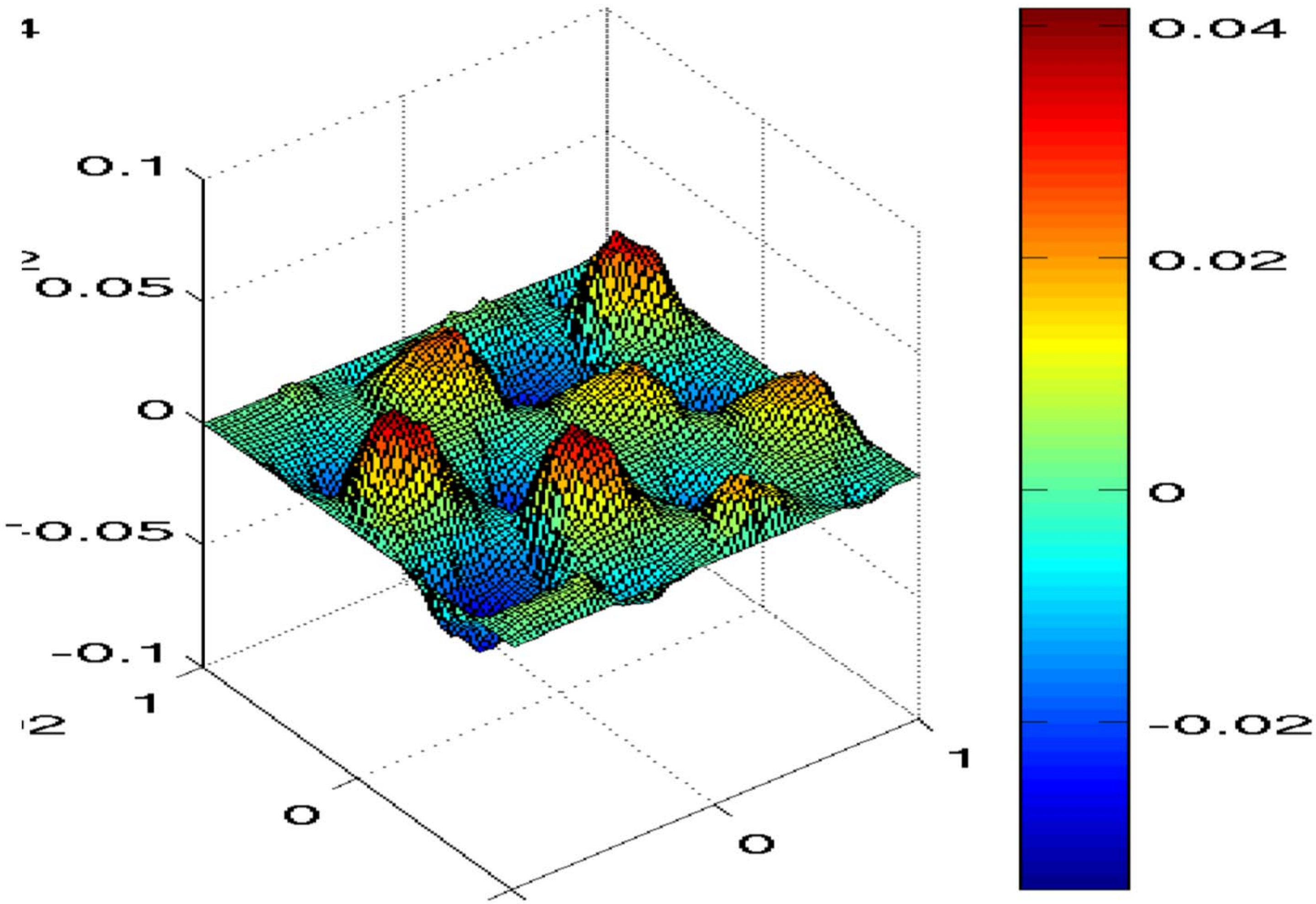


ode15s with $dt = 1/1280$

$$g(x,t) = \sin(2\pi(t + x_1))\cos(2\pi(t + x_2))$$

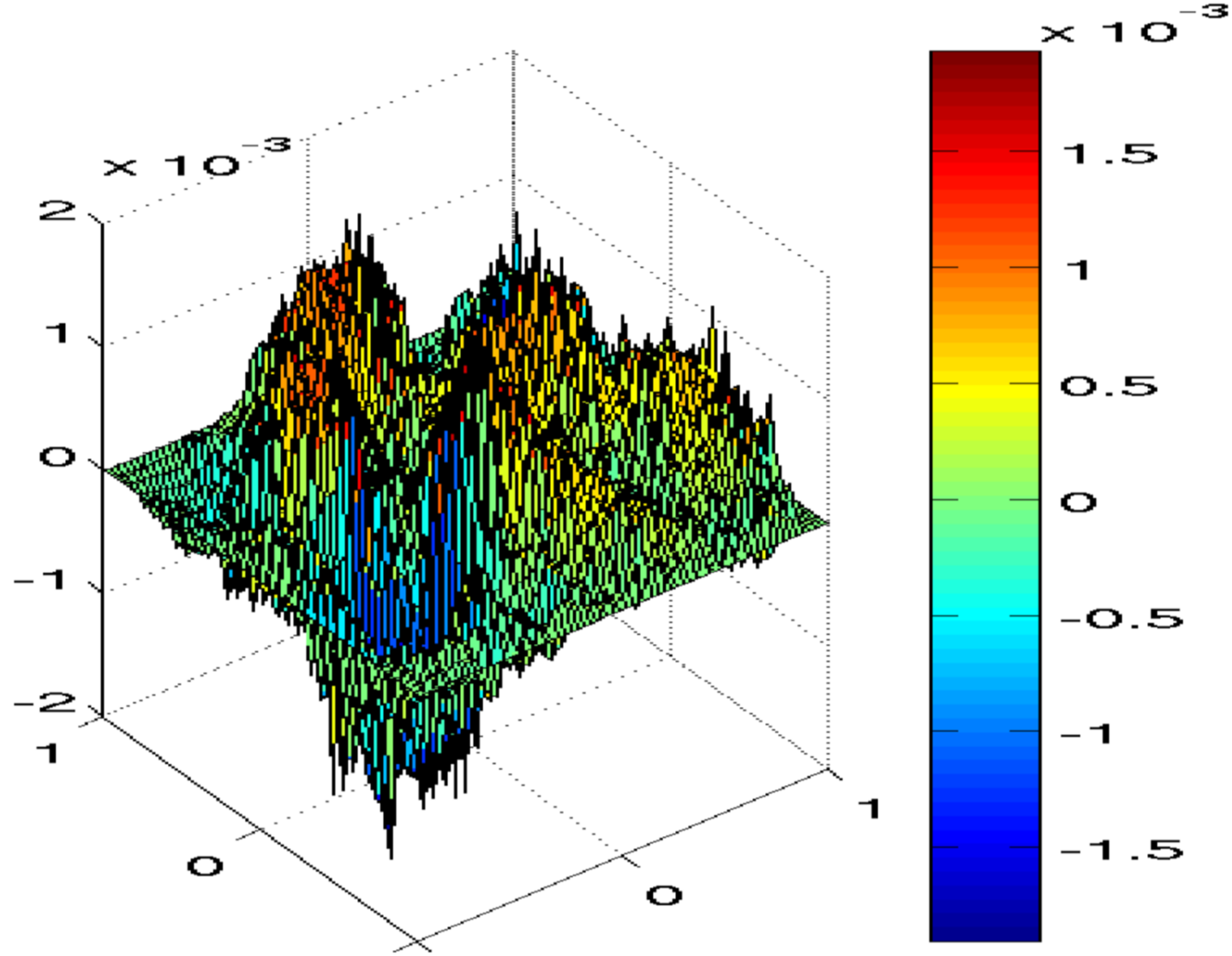
$$u(x,0) = \sin(2\pi x_1)\cos(2\pi x_2)$$

$\bar{u}$ at $T = 1$



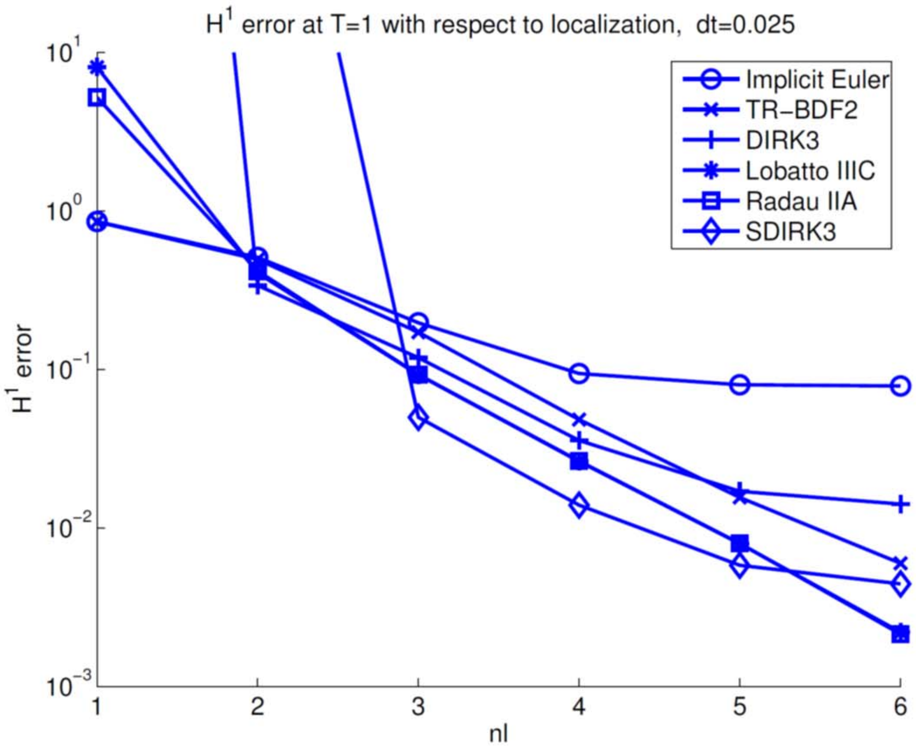
DIRK3 with $nl = 3$ and $\Delta t = 0.05$

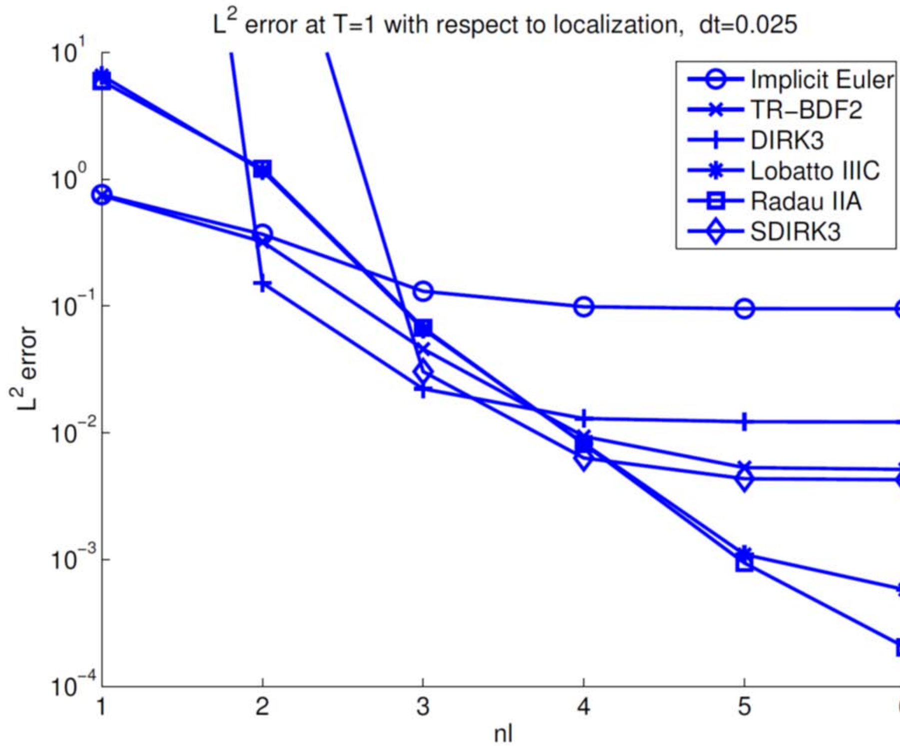
$u - \bar{u}$ at $T = 1$



$\frac{\Delta t}{dt} = 64$

Error vs localization ($T = 1$, $dt = 0.025$)





H^1 error $\quad\quad$ L^2 error

$N(nl)^d$ complexity

Joint with Zhang, Schäfer, Xie

Subspace Correction for Eigenvalue Problems

Combine the multilevel subspace correction scheme introduced by Lin & Xie with the gamblet subspace decomposition.
Setup: Coarse mesh space V_H, and a nested finite element space

$$V_H := V_1 \subset V_2 \subset \cdots \subset V_q =: V_h$$

One Correction Step: Suppose we have an eigenpair approximation $(\lambda_{l,k}, u_{l,k}) \in \mathbb{R} \times V_k$ for the exact eigenpair (λ, u).

1. Define the auxiliary source problem: find $\hat{u}_{l+1,k} \in V_k$ such that

$$a(\hat{u}_{l+1,k}, v_k) = \lambda_{l,k} b(u_{l,k}, v_k), \forall v_k \in V_k$$

   solve it in m multigrid iterations.
2. Define a new space $V_{H,k} = V_H + \mathrm{span}(\hat{u}_{l+1,k})$ and solve the eigenvalue problems: Find $(\lambda_{l+1,k}, u_{l+1,k}) \in \mathbb{R} \times V_{H,k}$, such that $b(u_{l+1,k}, u_{l+1,k}) = 1$ and

$$a(u_{l+1,k}, v_{H,k}) = \lambda_{l+1,k} b(u_{l+1,k}, v_{H,k}), \forall v_{H,k} \in V_{H,k}$$

Full multigrid scheme

1. Solve the coarse level eigenvalue problem in V_H: Find $(\lambda_H, u_H) \in \mathbb{R} \times V_H$, such that

$$a(u_H, v_H) = \lambda_H b(u_H, v_H), \forall v_H \in V_H$$

2. Multilevel iteration, for $k = 2, \cdots, q$, do
   - set $u_{0,k} = u_{k-1}$.
   - Do multigrid iteration

$$(\lambda_{l+1,k}, u_{l+1,k}) = \text{EigenMG}(V_H, \lambda_{l,k}, u_{l,k}, V_k, m), \text{for } l = 0, \cdots, p-1.$$

   - set $\lambda_k = \lambda_{p,k}$, and $u_k = u_{p,k}$.

   end do.

Convergence Rate

Convergence rate depends on the following parameters:

$$\delta_h(\lambda) := \sup_{\omega \in M(\lambda)} \inf_{v_h \in V_h} \|\omega - v_h\|_a.$$

and

$$\eta_a(h) := \sup_{f \in L^2(\Omega), \|f\|_b = 1} \inf_{v_h \in V_h} \|Tf - v_h\|_a.$$

where $T : L^2(\Omega) \to V$ is defined as

$$a(Tf, v) = b(f, v), \quad \forall f \in L^2(\Omega) \text{ and } \forall v \in V.$$

and
$M(\lambda) = \{\omega \in H_0^1(\Omega) : \omega \text{ is an eigenvector corresponding to } \lambda \text{ and } b(\omega, \omega) = 1\}$.

Gamblet provides a rigorously justified optimal bound for spaces V_h such that $\delta_h = O(h)$ and $\eta_a(h) = O(h)$.

Numerical Results: SPE10

Choose $\Omega = [-1,1] \times [-1,1]$, $H = 2/4$, $q = 6$, calculate the first 6 eigenvalues λ using gamblet based multigrid and bilinear interpolation based multigrid ($m = p = 2$), SPE10.

Relative errors for gamblet based multigrid

level	$\lambda_k^1 - \lambda^1$	$\lambda_k^2 - \lambda^2$	$\lambda_k^3 - \lambda^3$	$\lambda_k^4 - \lambda^4$	$\lambda_k^5 - \lambda^5$	$\lambda_k^6 - \lambda^6$
k=2	1.3299e-2	3.0490e-2	1.3287e-1	1.9856e-1	1.7584e-1	3.2068e-1
k=3	2.9030e-3	4.3332e-3	1.4580e-2	2.6290e-2	1.4929e-2	4.5882e-2
k=4	1.2450e-3	1.0300e-3	1.7021e-3	4.2904e-3	2.1925e-3	5.5244e-3
k=5	3.8248e-4	2.4465e-4	2.1443e-4	4.5721e-4	3.1176e-4	8.8591e-4
k=6	4.8006e-10	4.7992e-10	2.5963e-7	1.8019e-5	1.1164e-6	9.8976e-5

Relative errors for geometric multigrid

k=2	5.5962e-1	7.6254e-1	1.4811e0	1.5747e0	2.2069e0	2.7635e0
k=3	3.7138e-1	5.8835e-1	1.1421e0	1.1155e0	1.4946e0	1.9345e0
k=4	3.6455e-1	5.3818e-1	8.6044e-1	1.0038e0	1.4564e0	1.5450e0
k=5	3.6204e-1	5.0609e-1	7.0450e-1	9.8467e-1	1.4357e0	1.4290e0
k=6	3.5602e-1	5.0429e-1	6.9040e-1	9.7660e-1	1.4217e0	1.4122e0

Numerical Results: Random Chekerboard

Choose $\Omega = [-1,1] \times [-1,1]$, $H = 2/4$, $q = 6$, calculate the first 6 eigenvalues λ using gamblet based multigrid and bilinear interpolation based multigrid ($m = p = 2$), random checkerboard.

Relative errors for gamblet based multigrid

level	$\lambda_k^1 - \lambda^1$	$\lambda_k^2 - \lambda^2$	$\lambda_k^3 - \lambda^3$	$\lambda_k^4 - \lambda^4$	$\lambda_k^5 - \lambda^5$	$\lambda_k^6 - \lambda^6$
k=2	7.1092e-3	4.9591e-3	4.5251e-3	1.0614e-2	1.2118e-2	1.4976e-2
k=3	6.4277e-3	2.3379e-3	2.0046e-3	2.4096e-2	2.7884e-2	2.2965e-2
k=4	4.6439e-3	1.6492e-3	1.2202e-3	1.0138e-3	1.1059e-3	9.3171e-3
k=5	2.7618e-3	9.1602e-4	6.3657e-3	4.8176e-4	4.7764e-4	3.9240e-4
k=6	8.0415e-11	1.057e-10	5.9263e-11	3.9927e-10	5.545e-10	1.3302e-8

Relative errors for geometric multigrid

k=2	2.6698e0	2.3886e1	2.2749e0	2.6224e0	2.9668e0	2.5762e0
k=3	2.5627e0	2.3037e1	2.1802e0	2.5191e0	2.8135e0	2.4229e0
k=4	2.0945e0	1.8806e1	1.8073e0	2.0624e0	2.2931e0	1.9413e0
k=5	3.8291e-1	4.0754e-1	4.2985e-1	4.9242e-1	5.0825e-1	4.4409e-1
k=6	3/4186e-3	3.9342e-3	2.5185e-3	1.0877e-2	9.7022e-3	6.5403e-3

Thank you

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