Computational Information Games
A minitutorial Part II

Houman Owhadi

ICERM June 5, 2017

[H. Owhadi and C. Scovel. Universal Scalable Robust Solvers from Computational Information Games and fast eigenspace adapted Multiresolution Analysis 2017. arXiv:1703.10761]

DARPA EQUiPS / AFOSR award no FA9550-16-1-0054
(Computational Information Games)

Question

Can we design a linear solver with some degree of universality? (that could be applied to a large class of linear operators)

Motivation

There are (nearly) as many linear solvers as linear systems.

Number of google scholar references to "linear solvers": 447,000

Not clear that this can be done

"*Of course no one method of approximation of a 'linear operator' can be universal.*"

[Sard, 1967. Optimal approximation. Journal of Functional Analysis]

Arthur Sard (1909-1980)

$$\begin{cases} -\operatorname{div}(a\nabla u) = g, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

Multigrid Methods

Multigrid: [Fedorenko, 1961, Brandt, 1973, Hackbusch, 1978]

Multiresolution/Wavelet based methods

[Brewster and Beylkin, 1995, Beylkin and Coult, 1998, Averbuch et al., 1998

[Beylkin, Coifman, Rokhlin, 1992] [Engquist, Osher, Zhong, 1992]

[Alpert, Beylkin, Coifman, Rokhlin, 1993]

[Cohen, Daubechies, Feauveau. 1992]

[Bacry, Mallat, Papanicolaou. 1993]

- **Linear complexity with smooth coefficients**

Problem **Severely affected by lack of smoothness**

Robust/Algebraic multigrid

[Mandel et al., 1999,Wan-Chan-Smith, 1999,
Xu and Zikatanov, 2004, Xu and Zhu, 2008], [Ruge-Stüben, 1987]
[Panayot - 2010]

Stabilized Hierarchical bases, Multilevel preconditioners

[Vassilevski - Wang, 1997, 1998]

[Panayot - Vassilevski, 1997]

[Chow - Vassilevski, 2003]

[Aksoylu- Holst, 2010]

- Some degree of robustness

Low Rank Matrix Decomposition methods

Fast Multipole Method: [Greengard and Rokhlin, 1987]

Hierarchical Matrix Method: [Hackbusch et al., 2002]

[Bebendorf, 2008]:

$$N \ln^{2d+8} N \text{ complexity}$$

To achieve grid-size accuracy in L^2-norm

Hierarchical numerical homogenization method

[H. Owhadi, Multigrid with rough coefficients and Multiresolution operator decomposition from Hierarchical Information Games. SIAM Review, 2017]

First Solve $\mathcal{O}(N \ln^{3d} N)$ Subsequent solves $\mathcal{O}(N \ln^{d+1} N)$

To achieve grid-size accuracy in H^1-norm

$$Ax = b$$

Sparse matrix Laplacians

Sparsified Cholesky and Multigrid Solvers for Connection Laplacians: [Kyng, Lee, Peng, Sachdeva, Spielman , 2016]

Approximate Gaussian Elimination: [Kyng and Sachdeva, 2016]

$$N \, \mathrm{polylog}(N) \text{ complexity}$$

Structured sparse matrices (SDD matrices)

Graph sparsification: [Spielman and Teng , 2004]

Diagonally dominant linear systems: [Spielman and Teng , 2014]

[Koutis, Miller, Gary and Peng , 2014]

[Cohen, Kyng, Miller, Pachocki, Peng, Rao, and Xu, 2014]

[Kelner, Orecchia, Sidford, Zhu, 2013]

The problem

$\mathcal{T}$: Continuous linear bijection

$$\mathcal{B} \xrightarrow{\quad \mathcal{T} \quad} \mathcal{B}^*$$

We want to approximate $\mathcal{T}^{-1}$ and all its eigen-subpaces in near-linear complexity

For $u, v \in \mathcal{B}$,

- $[\mathcal{T}u, v] = [\mathcal{T}v, u]$,
- $[\mathcal{T}u, u] \geq 0$

$$\|u\|^2 := [\mathcal{T}u, u]$$

$(\mathcal{B}, \|\cdot\|)$: separable Banach space

Example

$$\begin{cases} -\operatorname{div}(a\nabla u) = g, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

$$\mathcal{T} = -\operatorname{div}(a\nabla \cdot)$$

$$(H_0^1(\Omega), \|\cdot\|_{H_0^1(\Omega)}) \xrightarrow{-\operatorname{div}(a\nabla\cdot)} (H^{-1}(\Omega), \|\cdot\|_{H^{-1}(\Omega)})$$

$$\mathcal{B} := H_0^1(\Omega)$$

$$\|u\|^2 := \int_\Omega (\nabla u)^T a \nabla u$$

Example

$$\mathcal{L}u = g$$

$\mathcal{L}$: arbitrary continuous linear bijection

$$\left(H_0^s(\Omega), \|\cdot\|_{H_0^s(\Omega)}\right) \xrightarrow{\quad\mathcal{L}\quad} \left(H^{-s}(\Omega), \|\cdot\|_{H^{-s}(\Omega)}\right)$$

$\mathcal{L}$: Symmetric and positive

- $[\mathcal{L}u, v] = [\mathcal{L}v, u]$,
- $[\mathcal{L}u, u] \geq 0$

$$\mathcal{B} := H_0^s(\Omega)$$

$$\mathcal{T} = \mathcal{L}$$

$$\|u\|^2 := [\mathcal{L}u, u]$$

Example

$$\mathcal{L}u = g \iff \mathcal{L}^*\mathcal{L}u = \mathcal{L}^*g$$

$\mathcal{L}$: arbitrary continuous linear bijection

$$\left(H_0^s(\Omega), \|\cdot\|_{H_0^s(\Omega)}\right) \xrightarrow{\mathcal{L}} \left(L^2(\Omega), \|\cdot\|_{L^2(\Omega)}\right)$$

$$\mathcal{B} := H_0^s(\Omega)$$

$$\mathcal{T} = \mathcal{L}^*\mathcal{L}$$

$$\|u\| := \|\mathcal{L}u\|_{L^2(\Omega)}$$

Example

$$Ax = b$$

A: $N \times N$ symmetric postive definite matrix

$$\mathcal{B} := \mathbb{R}^N$$

$$\mathcal{T} = A$$

$$\|x\|^2 := x^T A x$$

Example

$$Ax = b \Longleftrightarrow A^T A x = A^T b$$

A: $N \times N$ invertible matrix

$$\mathcal{B} := \mathbb{R}^N$$

$$\mathcal{T} = A^T A$$

$$\|x\|^2 := |Ax|^2$$

$$\mathcal{B} \xrightarrow{\quad \mathcal{T} \quad} \mathcal{B}^*$$

$$\|u\|^2 := [\mathcal{T}u, u]$$

Hierarchy of measurement functions

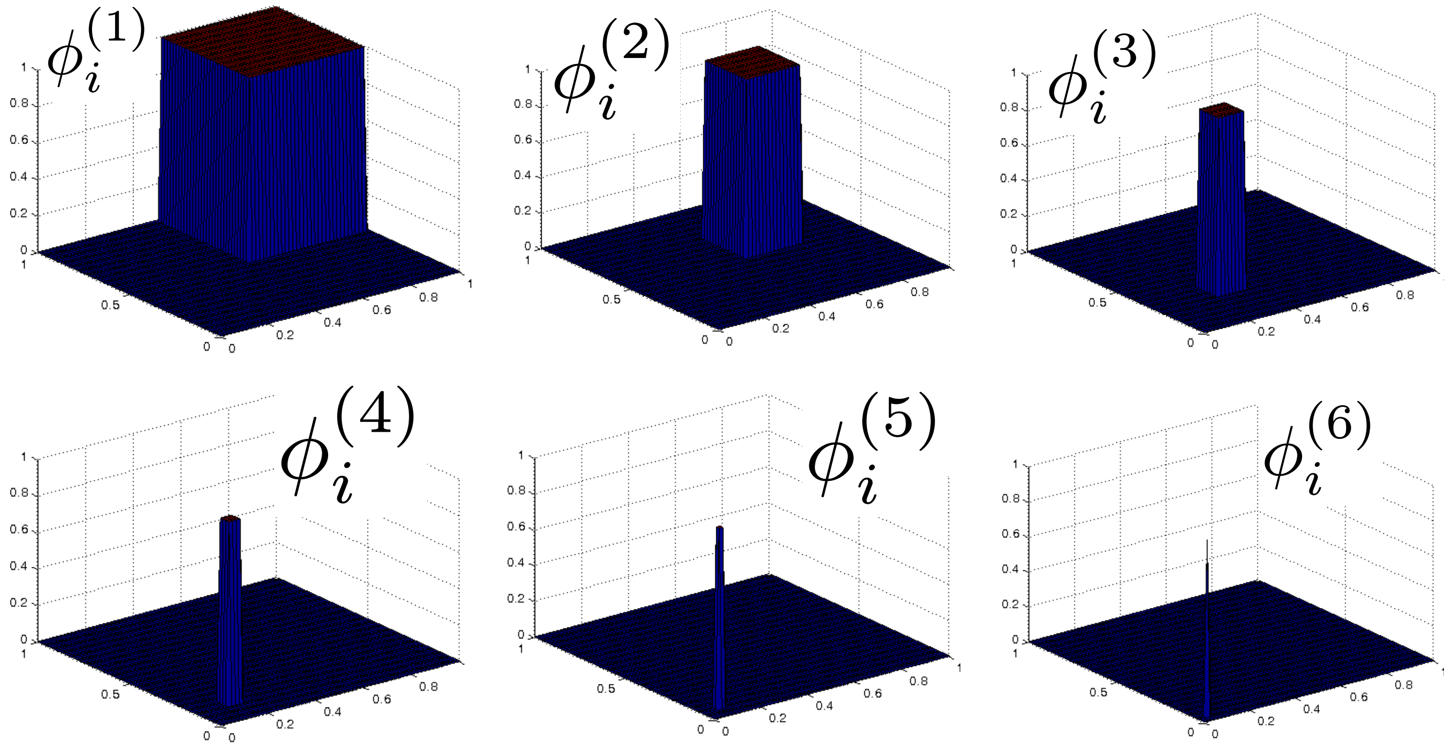
$$\phi_i^{(k)} \in \mathcal{B}^* \text{ with } k \in \{1, \ldots, q\}$$

$$\phi_i^{(k)} = \sum_j \pi_{i,j}^{(k,k+1)} \phi_j^{(k+1)}$$

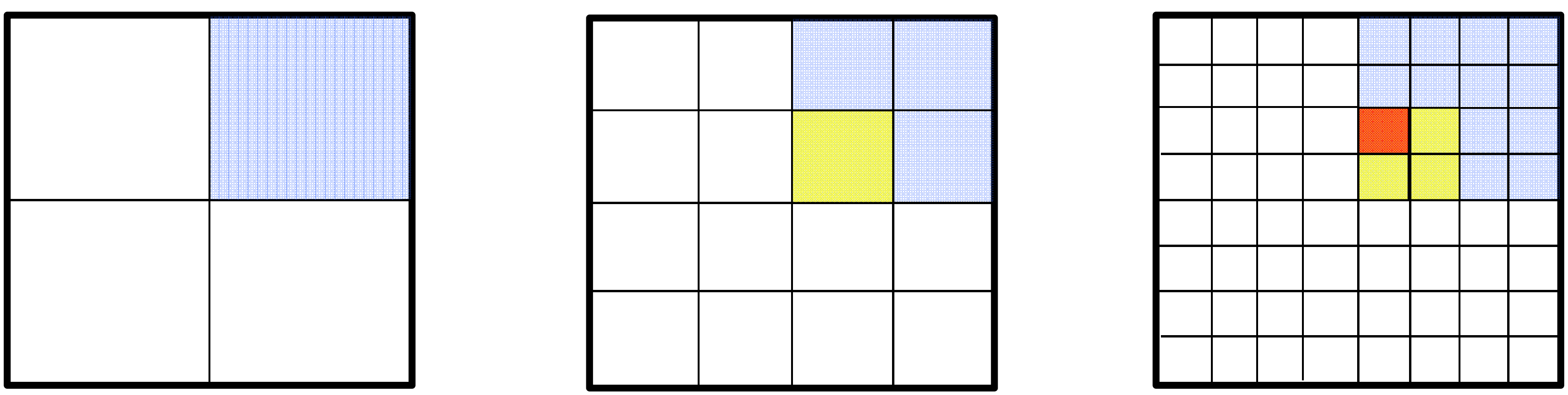
[Multigrid with rough coefficients and Multiresolution operator decomposition from Hierarchical Information Games. H. Owhadi. SIAM Review, 59(1), 99149, 2017. arXiv:1503.03467]

Example

$\mathcal{B} = H_0^s(\Omega)$



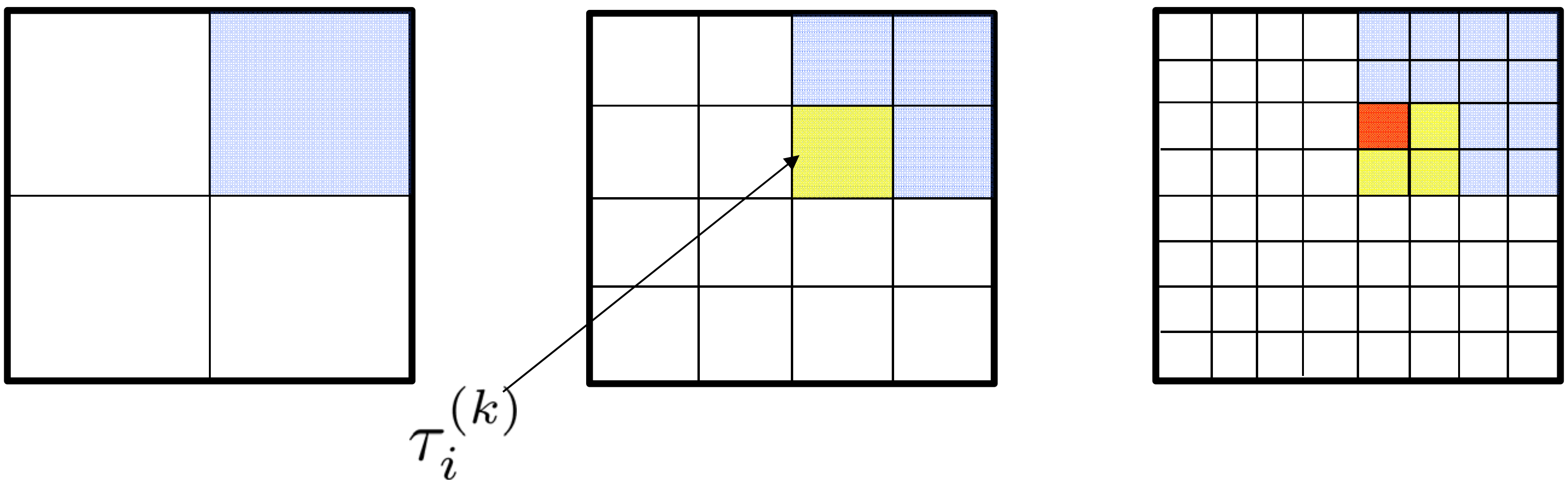
$\phi_i^{(k)}$: Weighted indicator functions of a hierarchical nested partition of Ω of resolution 2^{-k}



Example $\mathcal{B} = H_0^s(\Omega)$

$(\phi_{i,\alpha}^{(k)})_{\alpha \in \beth}$: orthonormal basis functions of $\mathcal{P}_{s-1}(\tau_i^{(k)})$

$\mathcal{P}_{s-1}(\tau_i^{(k)})$: polynomials of degree at most $s-1$

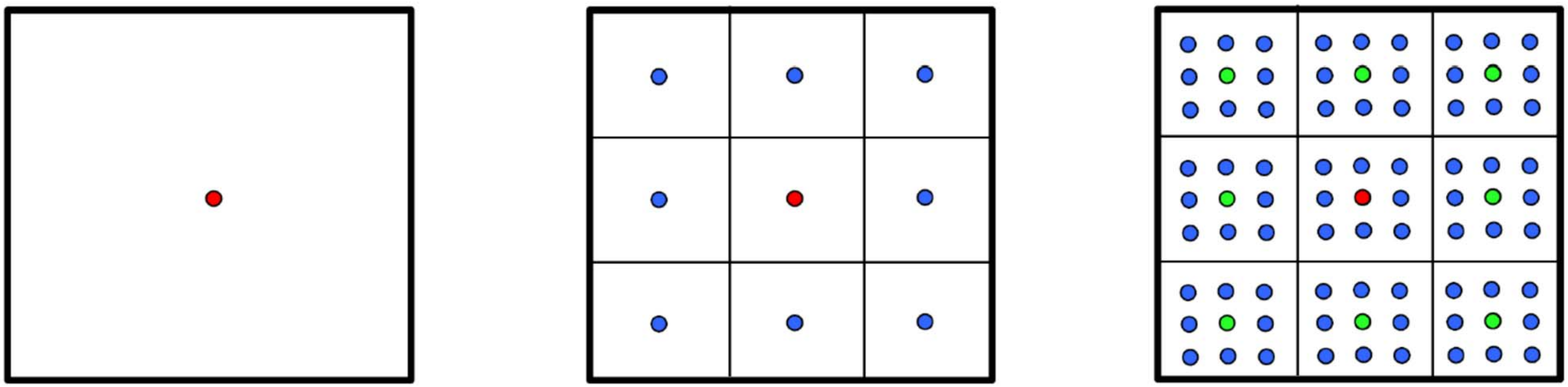


[H. Owhadi and C. Scovel. Universal Scalable Robust Solvers from Computational Information Games and fast eigenspace adapted Multiresolution Analysis 2017. arXiv:1703.10761]

T. Y. Hou and P. Zhang. Sparse operator compression of higher order elliptic operators with rough coefficients. *To appear*, 2017.

Example $\mathcal{B} = H_0^s(\Omega)$ $s > d/2$

$\phi_i^{(k)}$: Subsampled delta Dirac functions



[Schäfer, Sullivan, Owhadi. 2017]: **Compression, inversion, and approximate PCA of dense kernel matrices at near-linear computational complexity.**

[H. Owhadi and C. Scovel. Universal Scalable Robust Solvers from Computational Information Games and fast eigenspace adapted Multiresolution Analysis 2017. arXiv:1703.10761]

Player I

Chooses

$u \in \mathcal{B}$

Player II

Sees $[\phi_i^{(k)}, u],\ i \in \mathcal{I}_k$

Must predict

u and $[\phi_j^{(k+1)}, u],\ j \in \mathcal{I}_{k+1}$

Example

$\mathcal{B} = H_0^1(\Omega)$

Player I

Chooses

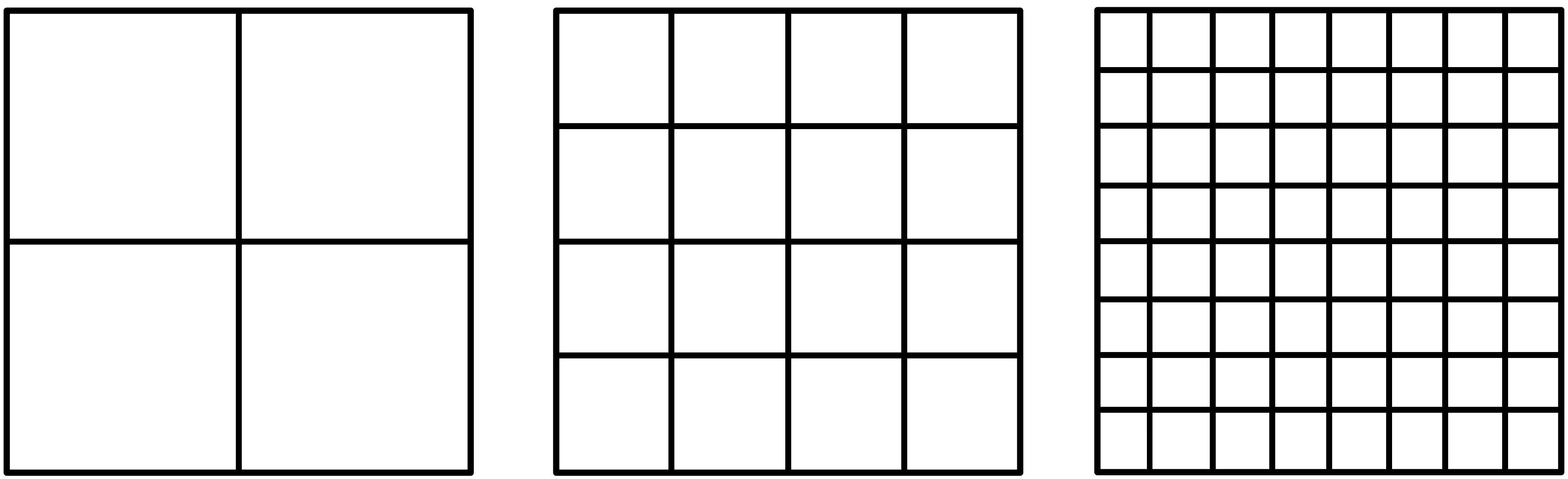
$u \in H_0^1(\Omega)$

Player II

Sees $\{\int_\Omega u\phi_i^{(k)},\ i \in \mathcal{I}_k\}$

Must predict

u and $\{\int_\Omega u\phi_j^{(k+1)},\ j \in \mathcal{I}_{k+1}\}$



Player II's bets

$$u^{(k)} := \mathbb{E}\left[\xi \middle| [\phi_i^{(k)}, \xi] = [\phi_i^{(k)}, u],\ i \in \mathcal{I}_k\right]$$

$$\mathcal{F}^{(k)} = \sigma([\phi_i^{(k)}, \xi],\ i \in \mathcal{I}_k)$$

$$\xi^{(k)} = \mathbb{E}[\xi | \mathcal{F}^{(k)}]$$

$\xi^{(k)}$: Martingale

$\xi^{(k)}$: Converging a.s.

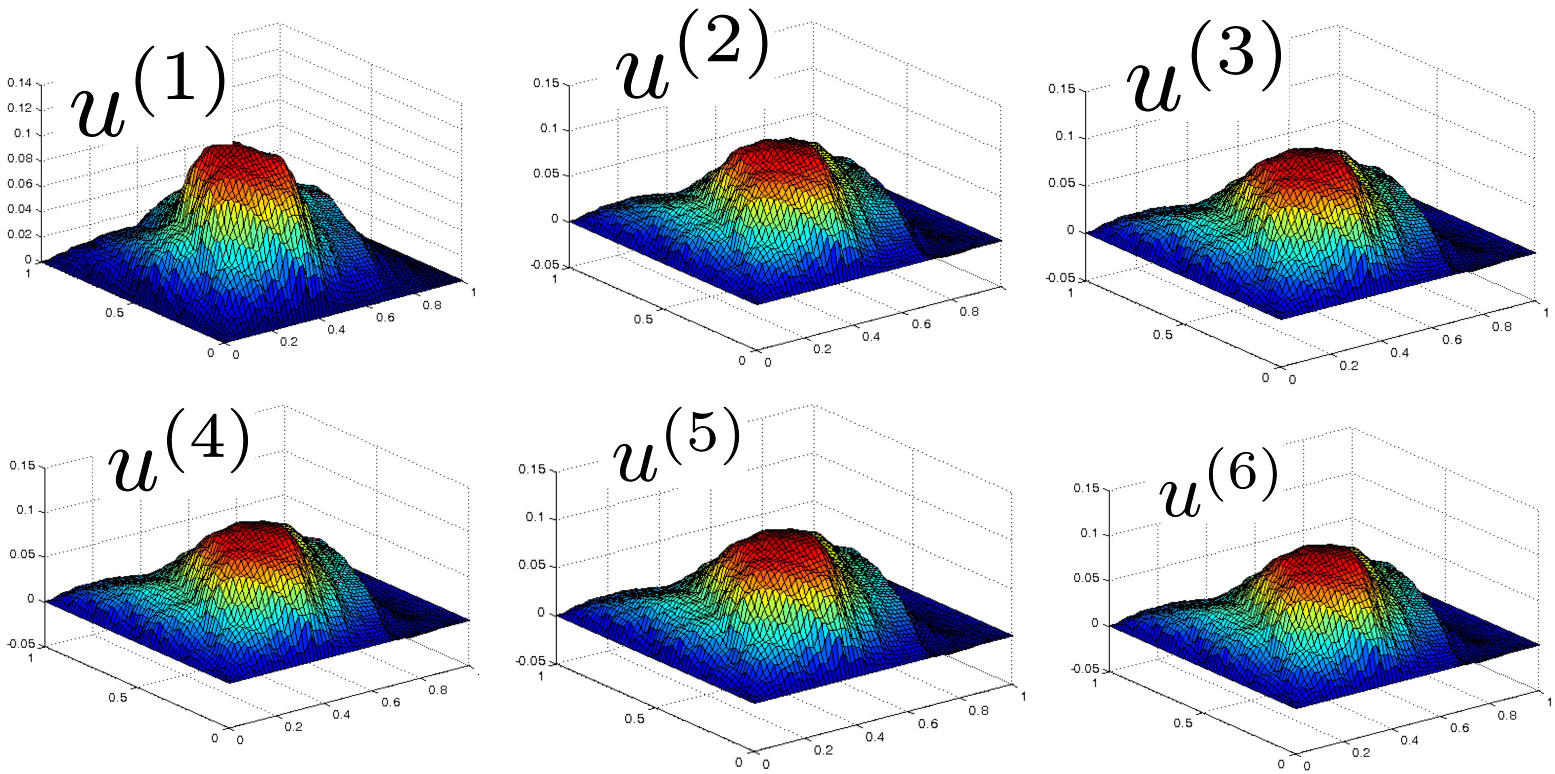
$\xi^{(k+1)} - \xi^{(k)}$: Uncorrelated (therefore independent)

Example

$$\mathcal{B} = H_0^1(\Omega)$$

$$\|u\|^2 = \int_\Omega (\nabla u)^T a \nabla u$$

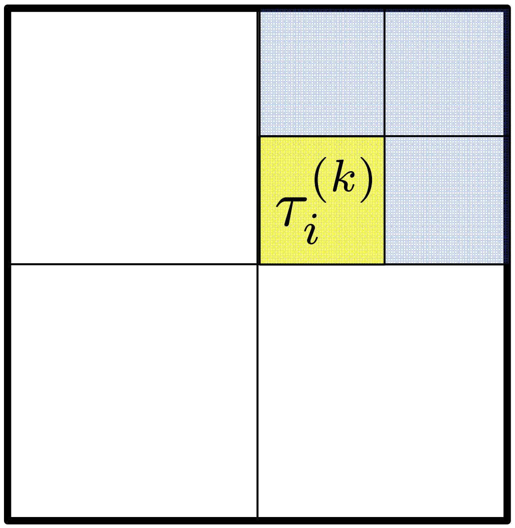
$$\begin{cases} -\operatorname{div}(a\nabla u) = g, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$



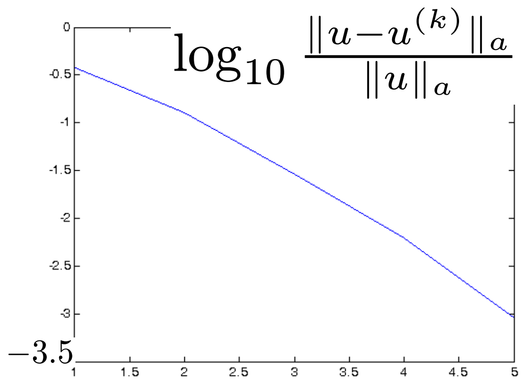
Accuracy of the recovery

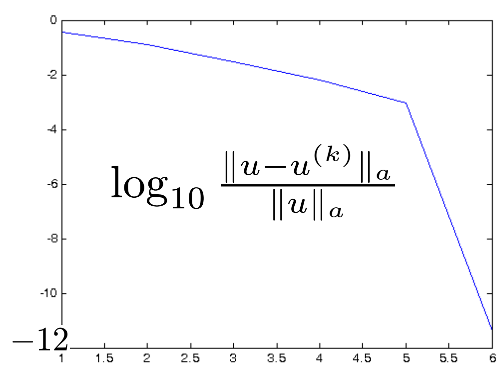
Theorem

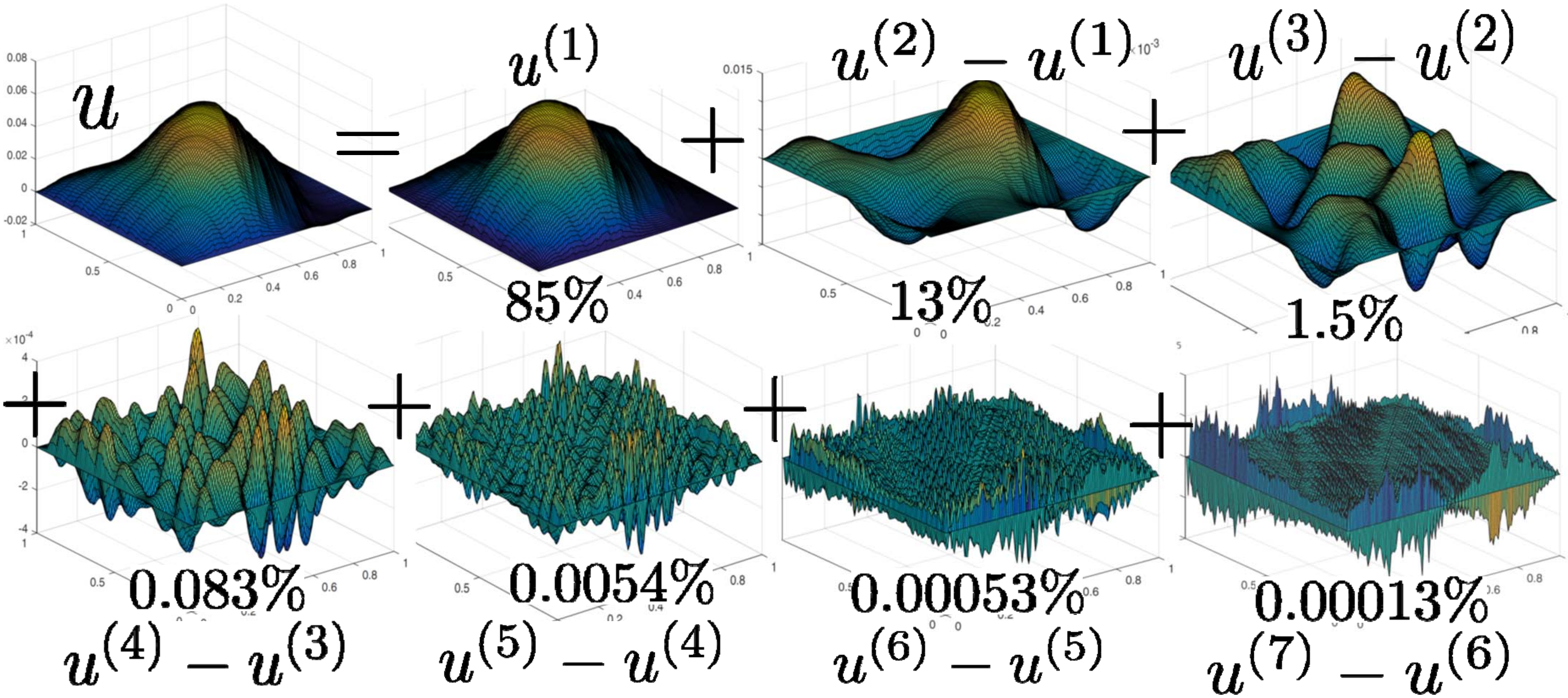
$$\|u - u^{(k)}\| \leq \frac{H^k}{\lambda_{\min}(a)} \|g\|_{L^2(\Omega)}$$



$$\phi_i^{(k)} = 1_{\tau_i^{(k)}} \qquad \mathrm{diam}(\tau_i^{(k)}) \leq H^k$$



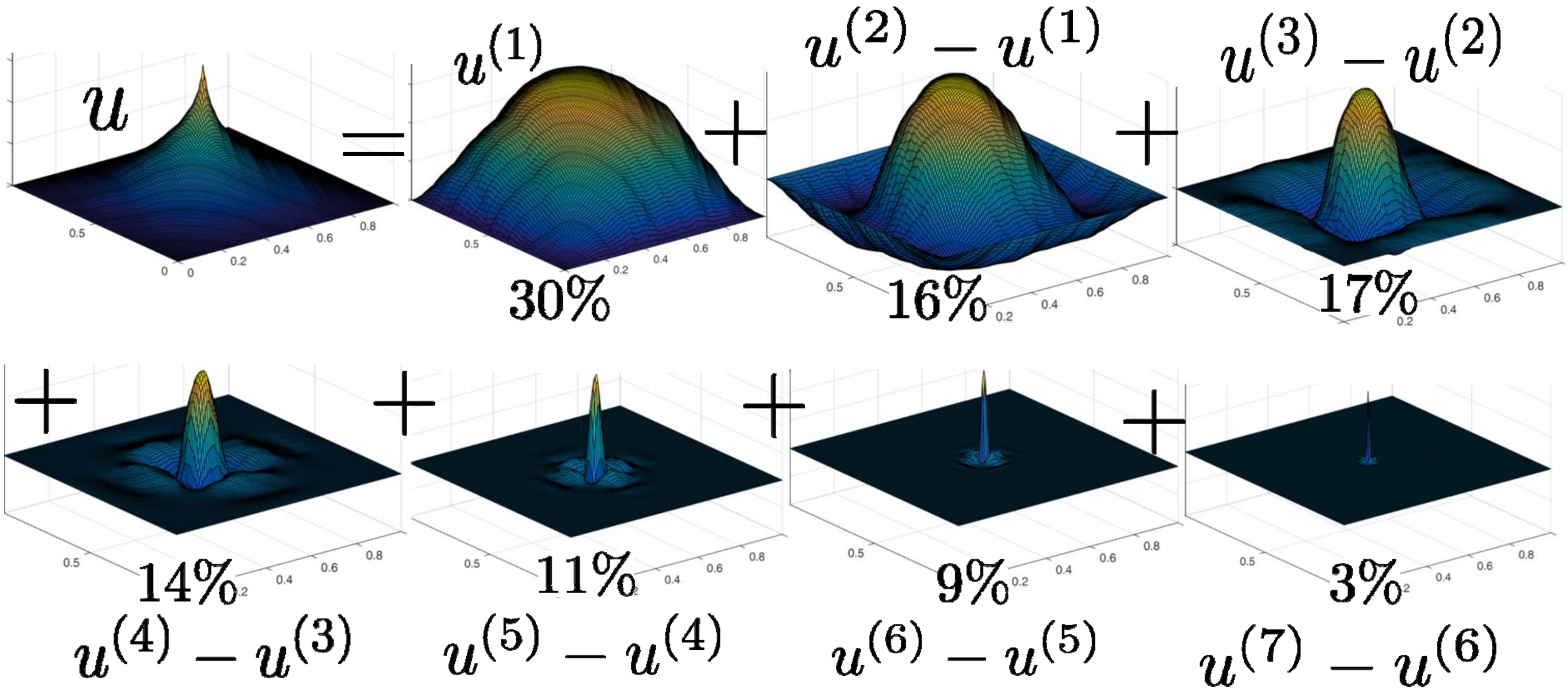




Energy content

$$\begin{cases} -\operatorname{div}(a\nabla u) = g, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \qquad g \in C^\infty(\Omega)$$

If r.h.s. is regular we don’t need to compute all subbands



Energy content

$$\begin{cases} -\operatorname{div}(a\nabla u) = g, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

$$g = \delta(x - x_0)$$

$$u^{(k)} = \sum_i [\phi_i^{(k)}, u]\, \psi_i^{(k)}$$

Gamblets

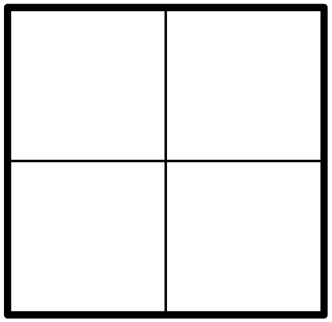
$$\psi_i^{(k)} = \mathbb{E}\big[\xi \big| [\phi_l^{(k)}, \xi] = \delta_{i,l},\, l \in \mathcal{I}_k\big]$$

Example

$$\mathcal{B} = H_0^1(\Omega)$$

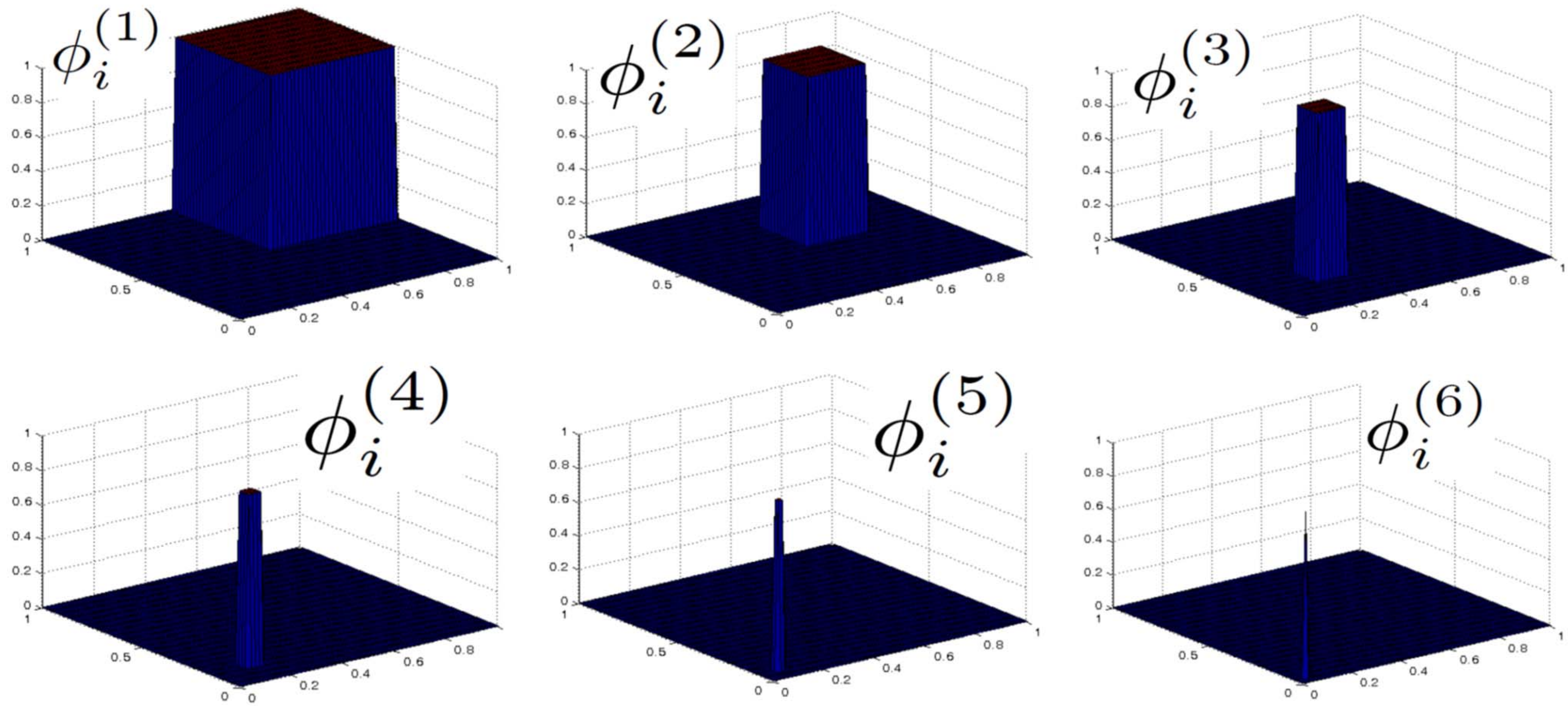
$$\|u\|^2 = \int_\Omega (\nabla u)^T a \nabla u$$

$$\phi_i^{(k-1)} = \sum_j \pi_{i,j}^{(k-1,k)} \phi_j^{(k)}$$

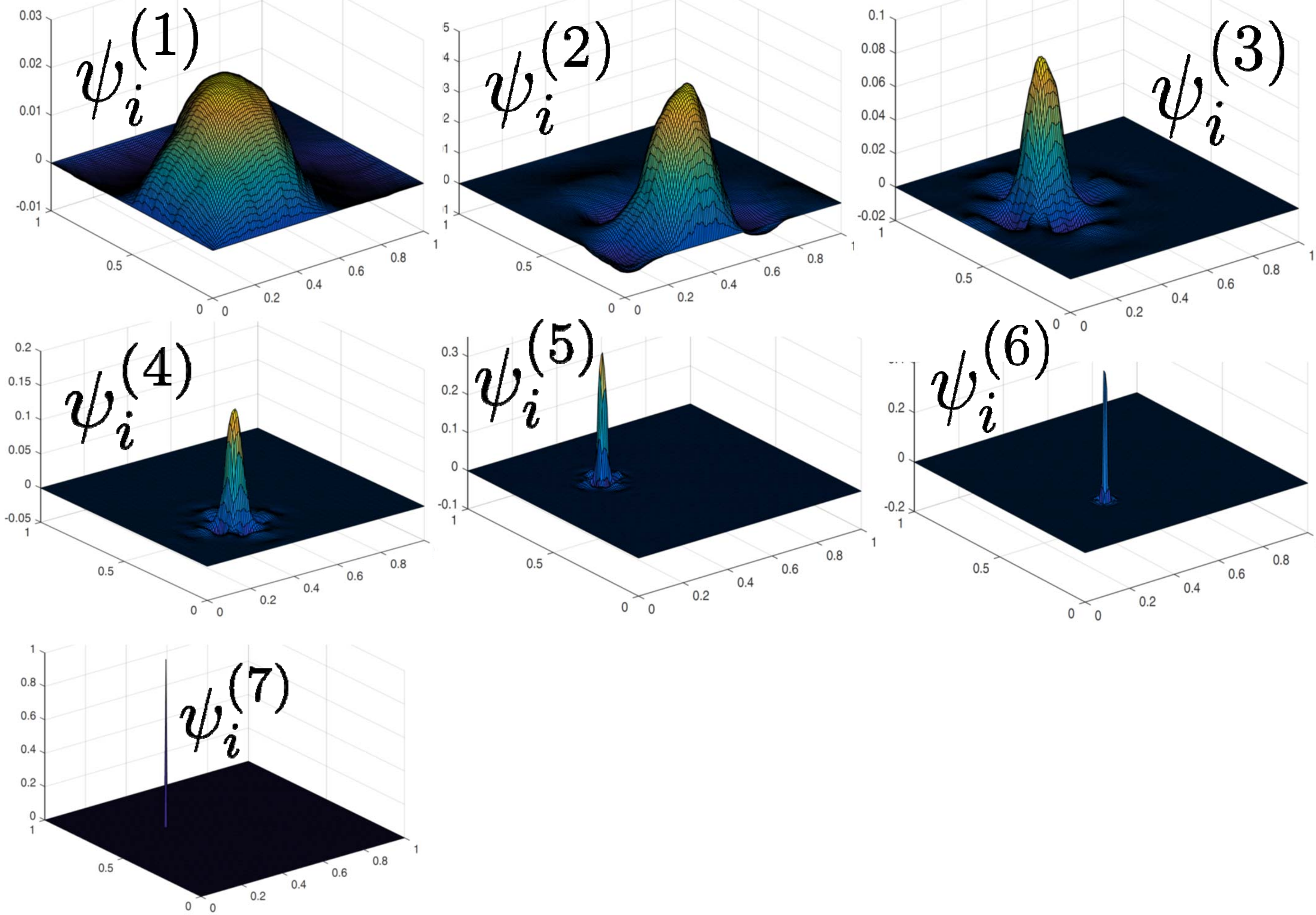


$\pi_{i,\cdot}^{(1,2)}$

0	0	1/2	1/2
0	0	1/2	1/2
0	0	0	0
0	0	0	0



Gamblets



$$\psi_i^{(k)} = \mathbb{E}\Big[\xi \Big| [\phi_l^{(k)}, \xi] = \delta_{i,l},\, l \in \mathcal{I}_k\Big]$$

Gamblets are nested

$$\psi_i^{(k)} = \sum_j R_{i,j}^{(k,k+1)} \psi_j^{(k+1)}$$

$$\psi_i^{(k)} = \mathbb{E}\Big[\mathbb{E}[\xi|\mathcal{F}_{k+1}] \Big| [\phi_l^{(k)}, \xi] = \delta_{i,l},\, l \in \mathcal{I}_k\Big]$$

$$\mathbb{E}[\xi|\mathcal{F}_{k+1}] = \sum_j [\phi_j^{(k+1)}, \xi]\, \psi_j^{(k+1)}$$

$$R_{i,j}^{(k,k+1)} = \mathbb{E}\Big[[\phi_j^{(k+1)}, \xi] \Big| [\phi_l^{(k)}, \xi] = \delta_{i,l},\, l \in \mathcal{I}_k\Big]$$

Interpolation/Prolongation operator

Player I

Player II

Chooses

$u \in \mathcal{B}$

Sees $[\phi_i^{(k)}, u]$, $i \in \mathcal{I}_k$

Must predict

$[\phi_j^{(k+1)}, u]$, $j \in \mathcal{I}_{k+1}$

Optimal bet of Player II on the value of $[\phi_j^{(k+1)}, u]$

$$\sum_i [\phi_i^{(k)}, u] R_{i,j}^{(k,k+1)}$$

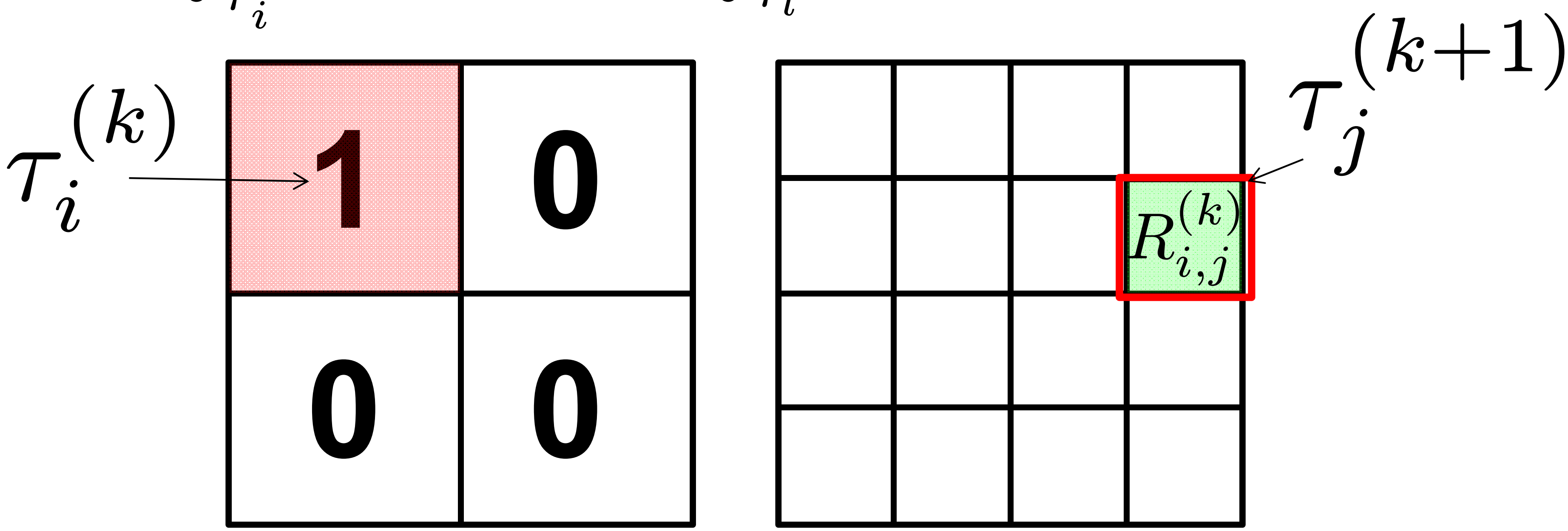
Example

$$\mathcal{B} = H_0^1(\Omega)$$

$$\begin{cases} -\operatorname{div}(a\nabla u) = g, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

$$R_{i,j}^{(k)} = \mathbb{E}\Big[\int_\Omega \xi(y)\phi_j^{(k+1)}(y)\,dy \Big| \int_\Omega \xi(y)\phi_l^{(k)}(y)\,dy = \delta_{i,l},\, l \in \mathcal{I}_k\Big]$$

$R_{i,j}^{(k)}$ Your best bet on the value of $\int_{\tau_j^{(k+1)}} u$ given the information that $\int_{\tau_i^{(k)}} u = 1$ and $\int_{\tau_l} u = 0$ for $l \neq i$



$$\mathcal{B} \xrightarrow{\mathcal{T}} \mathcal{B}^*$$

$$\|u\|^2 := [\mathcal{T}u, u]$$

Hierarchy of measurement functions

$$\phi_i^{(k)} \in \mathcal{B}^* \text{ with } k \in \{1, \ldots, q\}$$

$$\phi_i^{(k)} = \sum_j \pi_{i,j}^{(k,k+1)} \phi_j^{(k+1)}$$

Hierarchy of gamblets

$$\psi_i^{(k)} = \sum_{j \in \mathcal{I}^{(k)}} \Theta_{i,j}^{(k),-1} \mathcal{T}^{-1} \phi_j^{(k)}$$

$$\Theta_{i,j}^{(k)} := [\phi_i^{(k)}, \mathcal{T}^{-1} \phi_j^{(k)}]$$

Biorthogonal system

$$[\phi_j^{(k)}, \psi_i^{(k)}] = \delta_{i,j}$$

$$\mathfrak{V}^{(k)} := \mathrm{span}\{\psi_i^{(k)} \mid i \in \mathcal{I}^{(k)}\}$$

Theorem

The $\langle \cdot, \cdot \rangle$ orthogonal projection of $u \in \mathcal{B}$ onto $\mathfrak{V}^{(k)}$ is

$$u^{(k)} = \sum_{i \in \mathcal{I}^{(k)}} [\phi_i^{(k)}, u] \psi_i^{(k)}$$

Measurement functions are nested

$$\phi_i^{(k)} = \sum_j \pi_{i,j}^{(k,k+1)} \phi_j^{(k+1)}$$

Gamblets are nested

$$\psi_i^{(k)} = \sum_{j\in\mathcal{I}^{(k+1)}} R_{i,j}^{(k,k+1)} \psi_j^{(k+1)}$$

Orthogonalized gamblets

$$\chi_i^{(k)} := \sum_{j\in\mathcal{I}^{(k)}} W_{i,j}^{(k)} \psi_j^{(k)}$$

For $k \geq 2$ $\quad W^{(k)}$: $\mathcal{J}^{(k)} \times \mathcal{I}^{(k)}$ matrix such that
$\mathrm{Img}(W^{(k),T}) = \mathrm{Ker}(\pi^{(k-1,k)})$
and $W^{(k)}(W^{(k)})^T = J^{(k)}$

$$\chi_i^{(k)} := \sum_{j\in\mathcal{I}^{(k)}} W_{i,j}^{(k)} \psi_j^{(k)}$$

$t^{(1)} = l^{(1)} = r^{(1)} = i$

0	0	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
0	0	0	0
0	0	0	0
0	0	0	0

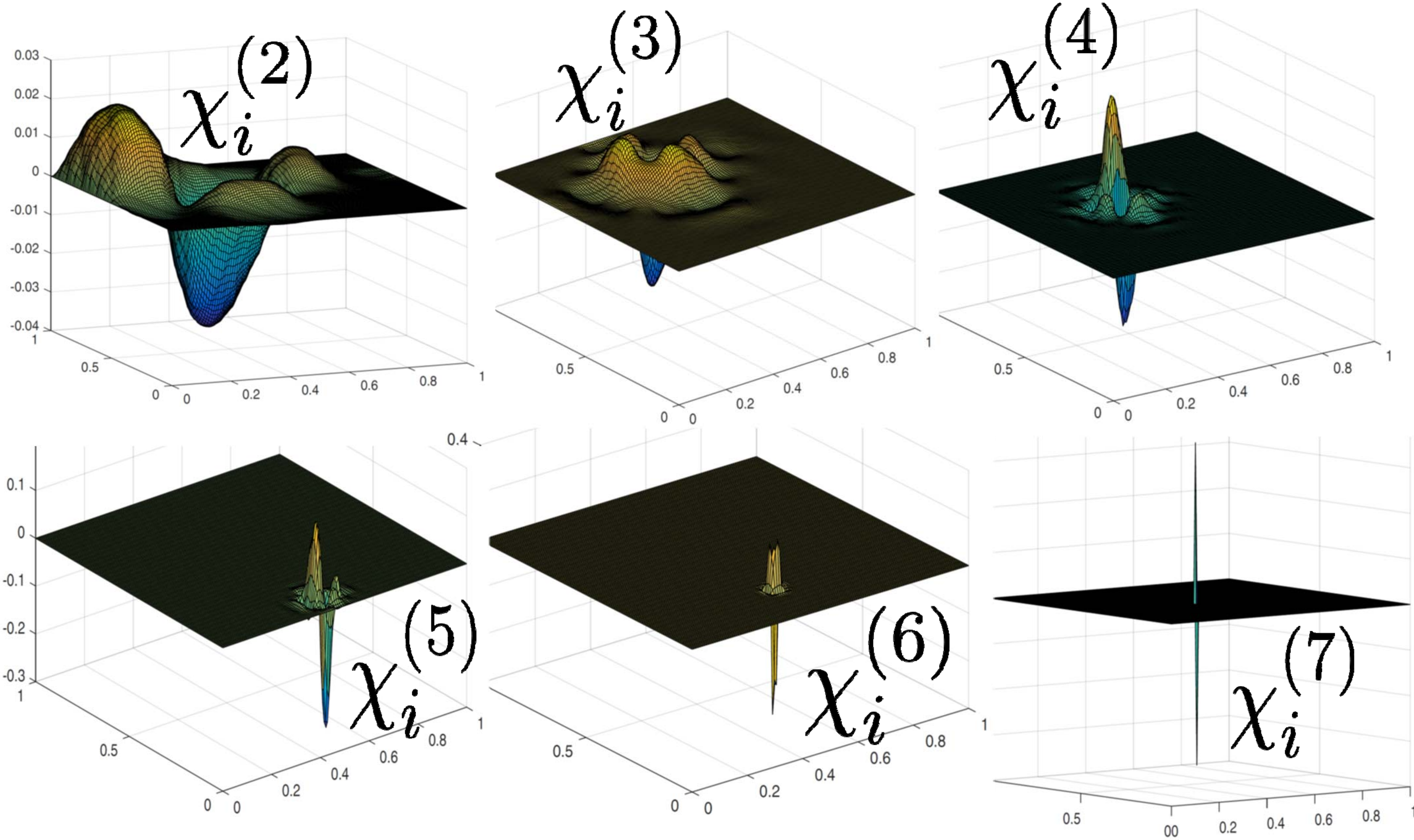
$W^{(2)}_{t,\cdot}$

0	0	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{6}}$
0	0	$-\frac{2}{\sqrt{6}}$	0
0	0	0	0
0	0	0	0

$W^{(2)}_{l,\cdot}$

0	0	$\frac{1}{\sqrt{12}}$	$\frac{1}{\sqrt{12}}$
0	0	$\frac{1}{\sqrt{12}}$	$-\frac{3}{\sqrt{12}}$
0	0	0	0
0	0	0	0

$W^{(2)}_{r,\cdot}$



Operator adapted MRA

$$\mathfrak{V}^{(k)} := \text{span}\{\psi_i^{(k)} \mid i \in \mathcal{I}^{(k)}\}$$

$$\mathfrak{W}^{(k)} := \text{span}\{\chi_i^{(k)} \mid i \in \mathcal{I}^{(k)}\}$$

Theorem

$$\mathfrak{V}^{(k)} = \mathfrak{V}^{(k-1)} \oplus \mathfrak{W}^{(k)}$$

$$\mathcal{B} = \mathfrak{V}^{(1)} \oplus \mathfrak{W}^{(2)} \oplus \mathfrak{W}^{(3)} \oplus \cdots$$

$u^{(k)} - u^{(k-1)}$: The $\langle \cdot, \cdot \rangle$ orthogonal projection of $u \in \mathcal{B}$ onto $\mathfrak{W}^{(k)}$

$$\mathcal{B} \xrightarrow{\mathcal{T}} \mathcal{B}^*$$

$$u \longrightarrow g$$

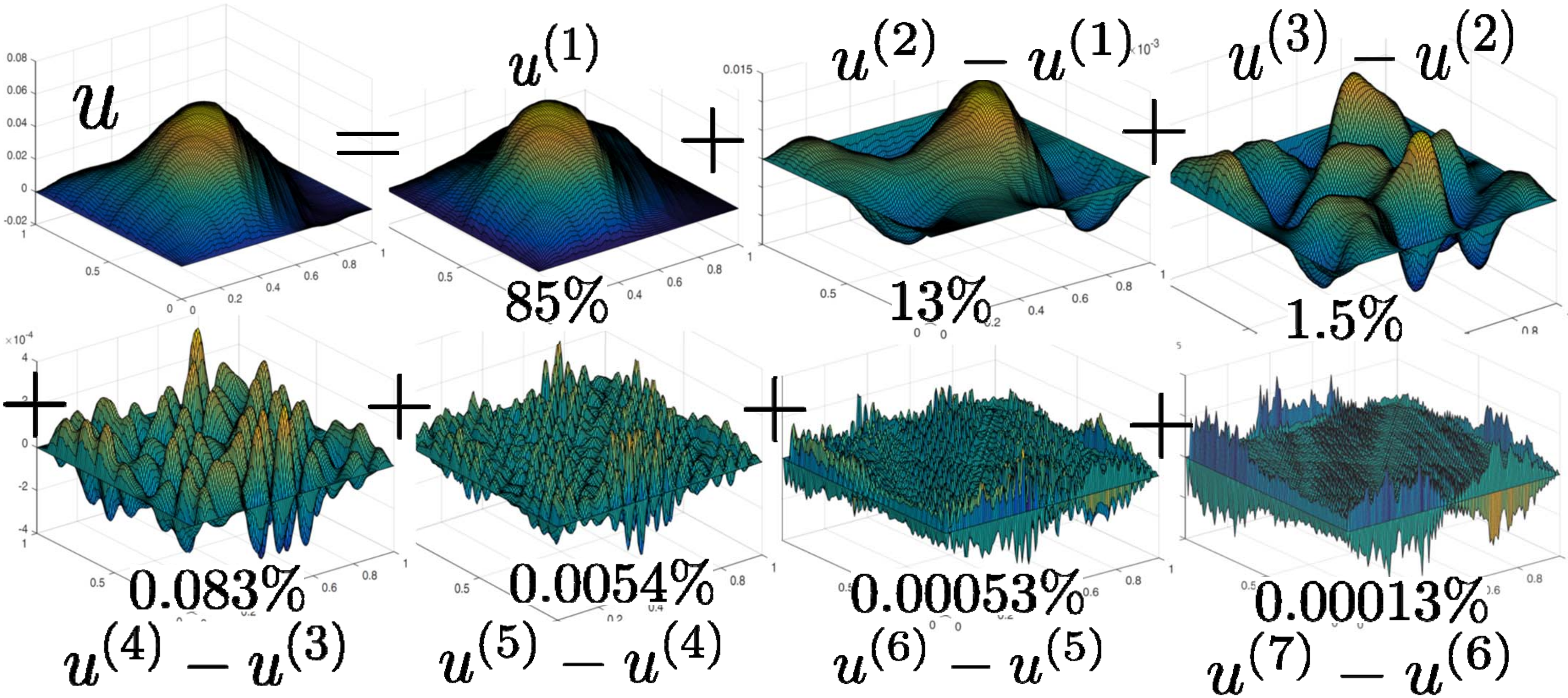
$$\mathcal{T} u = g$$

Theorem $u = u^{(1)} + \cdots + (u^{(k)} - u^{(k-1)}) + \cdots$

$$u^{(k)} - u^{(k-1)} = \sum_{i \in \mathcal{I}^{(k)}} w_i^{(k)} \chi_i^{(k)}$$

$$B^{(k)} w^{(k)} = g^{(k)}$$

$$g_i^{(k)} = [g, \chi_i^{(k)}] \qquad B_{i,j}^{(k)} = \langle \chi_i^{(k)}, \chi_j^{(k)} \rangle$$

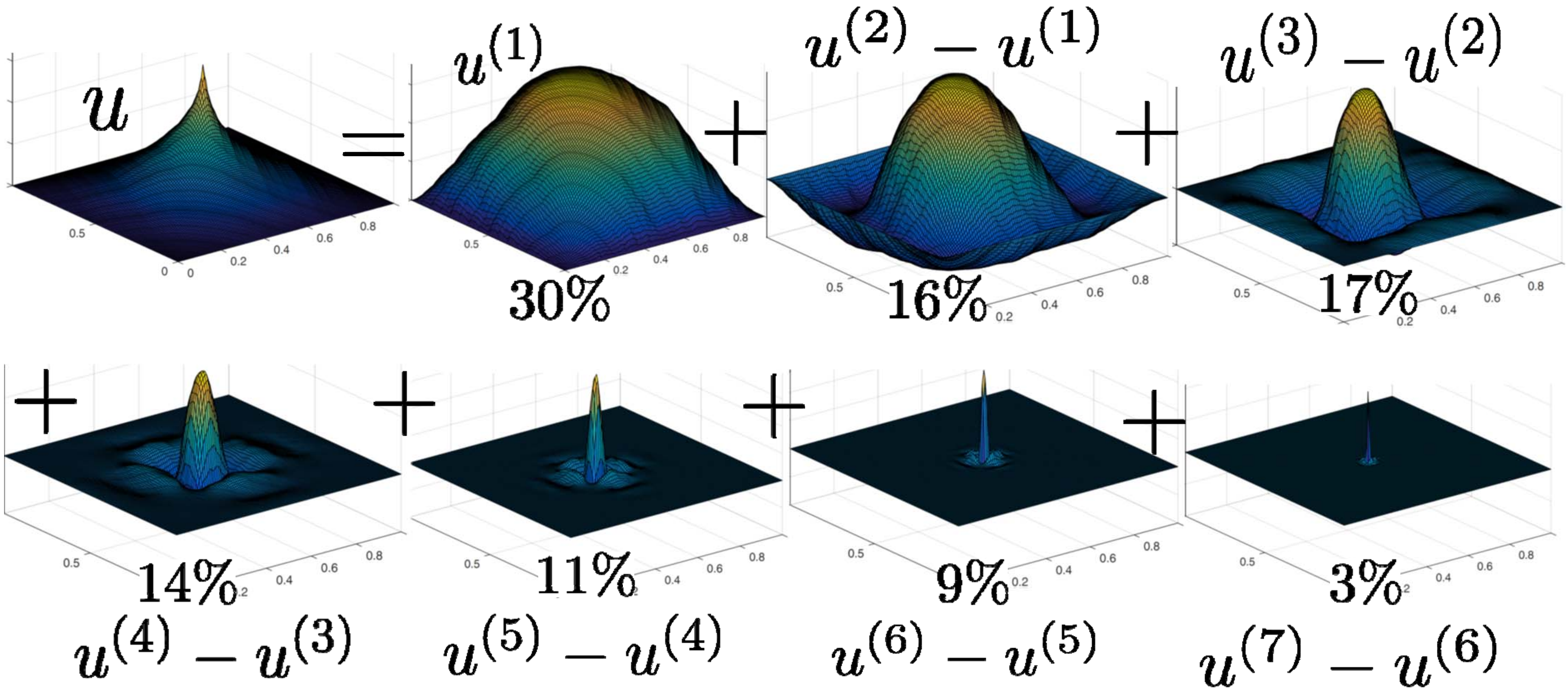


Energy content

$$\begin{cases} -\operatorname{div}(a\nabla u) = g, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

$$g \in C^\infty(\Omega)$$

If r.h.s. is regular we don't need to compute all subbands



Energy content

$$\begin{cases} -\operatorname{div}(a\nabla u) = g, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

$$g = \delta(x - x_0)$$

Operator adapted wavelets

First Generation Wavelets: Signal and imaging processing

[Mallat, 1989] [Daubechies, 1990]

[Coifman, Meyer, and Wickerhauser, 1992]

First Generation Operator Adapted Wavelets (shift and scale invariant)

[Cohen, Daubechies, Feauveau. Biorthogonal bases of compactly supported wavelets. 1992]

[Beylkin, Coifman, Rokhlin, 1992] [Engquist, Osher, Zhong, 1992]

[Alpert, Beylkin, Coifman, Rokhlin, 1993] [Jawerth, Sweldens, 1993]

[Dahlke, Weinreich, 1993] [Bacry, Mallat, Papanicolaou. 1993]

[Bertoluzza, Maday, Ravel, 1994] [Vasilyev, Paolucci, 1996]

[Dahmen, Kunoth, 2005] [Stevenson, 2009]

Lazy wavelets (Multiresolution decomposition of solution space)

[Yserentant. Multilevel splitting, 1986]

[Bank, Dupont, Yserentant. Hierarchical basis multigrid method. 1988]

Operator adapted wavelets

Second Generation Operator Adapted Wavelets

[Sweldens. The lifting scheme, 1998] [Dorobantu - Engquist. 1998]

[Vassilevski, Wang. Stabilizing the hierarchical basis, 1997]

[Carnicer, Dahmen, Peña, 1996] [Lounsbery, DeRose, Warren, 1997]

[Vassilevski, Wang. Stabilizing hierarchical basis, 1997-1998]

[Barinka, Barsch, Charton, Cohen, Dahlke, Dahmen, Urban, 2001]

[Cohen, Dahmen, DeVore, 2001] [Chiavassa, Liandrat, 2001]

[Dahmen, Kunoth, 2005] [Schwab, Stevenson, 2008]

[Sudarshan, 2005] [Engquist, Runborg, 2009] [Yin, Liandrat, 2016]

We want

1. **Scale-orthogonal wavelets with respect to operator scalar product (leads to block-diagonalization)**
2. **Operator to be well conditioned within each subband**
3. **Wavelets need to be localized (compact support or exp. decay)**

Eigenspace adapted MRA

$$A_{i,j}^{(k)} = \langle \psi_i^{(k)}, \psi_j^{(k)} \rangle \qquad B_{i,j}^{(k)} = \langle \chi_i^{(k)}, \chi_j^{(k)} \rangle$$

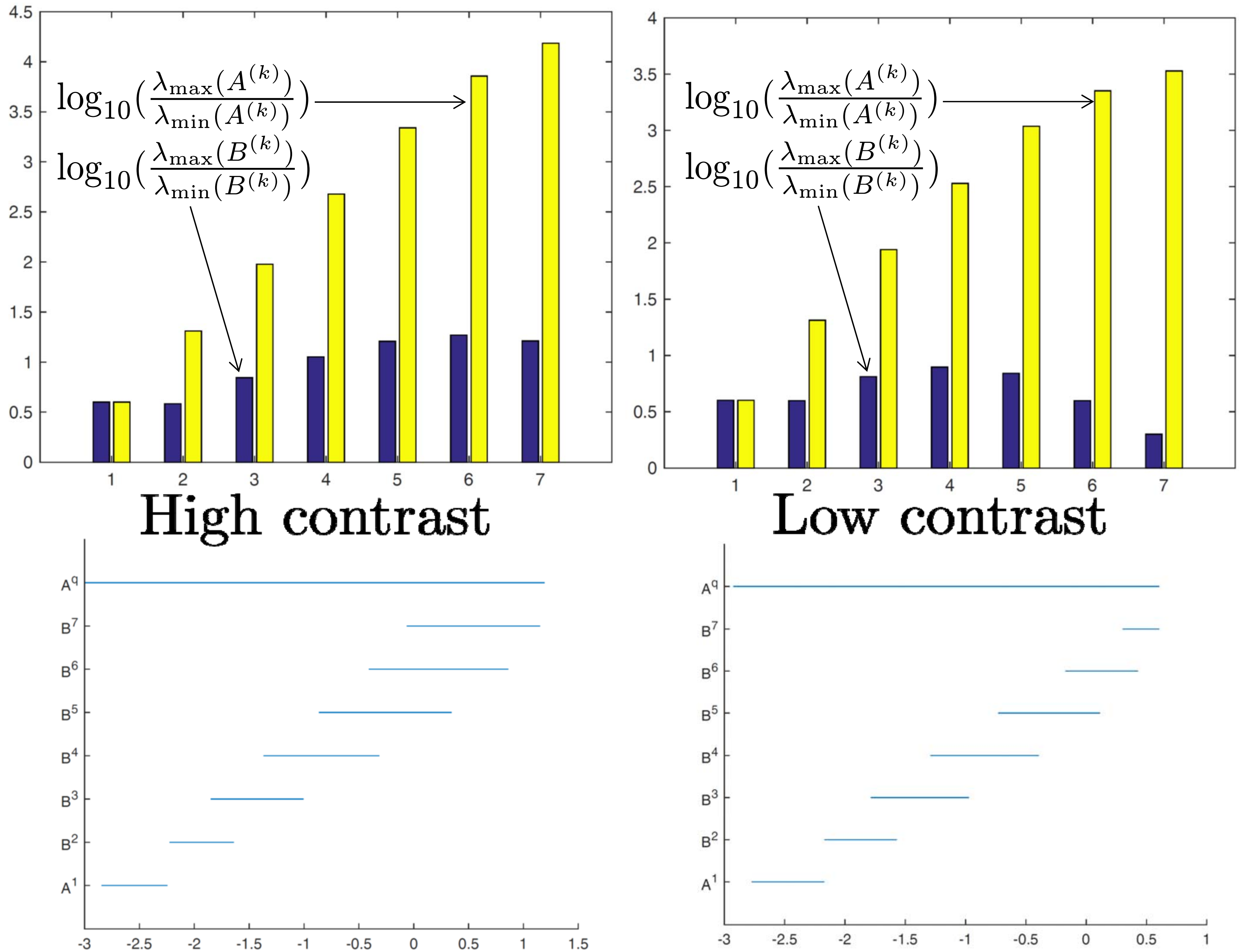
Theorem Under regularity of measurement functions

$$\frac{1}{C} H^{-2(k-1)} J^{(k)} \leq B^{(k)} \leq C H^{-2k} J^{(k)}$$

$$\boxed{\mathrm{Cond}(B^{(k)}) \leq C H^{-2}}$$

$$\frac{1}{C} I^{(1)} \leq A^{(1)} \leq C H^{-2} I^{(1)}$$

$$\mathrm{Cond}(A^{(1)}) \leq C H^{-2}$$



$\log_{10}(\frac{\lambda_{\max}(A^{(k)})}{\lambda_{\min}(A^{(k)})})$
$\log_{10}(\frac{\lambda_{\max}(B^{(k)})}{\lambda_{\min}(B^{(k)})})$
High contrast
Low contrast
A^q
B^7
B^6
B^5
B^4
B^3
B^2
A^1

Wannier functions

[Wannier. Dynamics of band electrons in electric and magnetic fields. 1962]

[Kohn. Analytic properties of Bloch waves and Wannier functions, 1959]

[Marzari, Vanderbilt. Maximally localized generalized Wannier functions for composite energy bands. 1997]

[E, Tiejun, Jianfeng. Localized bases of eigensubspaces and operator compression, 2010]

[Vidvuds, Lai, Caflisch, Osher, Compressed modes for variational problems in mathematics and physics, 2013]

[Owhadi, Multiresolution operator decomposition, SIREV 2017]

[Owhadi, Zhang, gamblets for hyperbolic and parabolic PDEs, 2016]

[Hou, Qin, Zhang, A sparse decomposition of low rank symmetric positive semi-definite matrices, 2016]

[Hou, Zhang, Sparse operator compression of elliptic operators. 2017]

$$(\mathcal{B}, \|\cdot\|) \xrightarrow{\qquad\qquad\mathcal{T}\qquad\qquad} (\mathcal{B}^*, \|\cdot\|_*)$$

Regularity Conditions

For some $H \in (0,1)$ and $C_\Phi > 0$

1. $|x| \leq C_\Phi H^{-k} \|\phi\|_*$
   for $\phi \in \{\sum_{i \in \mathcal{I}^{(k)}} x_i \phi_i^{(k)}\}$

2. $\|\phi\|_* \leq C_\Phi H^k |x|$
   for $\phi \in \{\sum_{i \in \mathcal{I}^{(k+1)}} x_i \phi_i^{(k+1)} \mid x \in \operatorname{Ker}(\pi^{(k,k+1)})\}$

Conditions are covariant under norm equivalence

Example $\mathcal{T} = \mathcal{L}$

$$(H_0^s(\Omega), \|\cdot\|_{H_0^s(\Omega)}) \xrightarrow{\quad\mathcal{L}\quad} (H^{-s}(\Omega), \|\cdot\|_{H^{-s}(\Omega)})$$

Regularity Conditions

For some $H \in (0,1)$ and $C_s > 0$

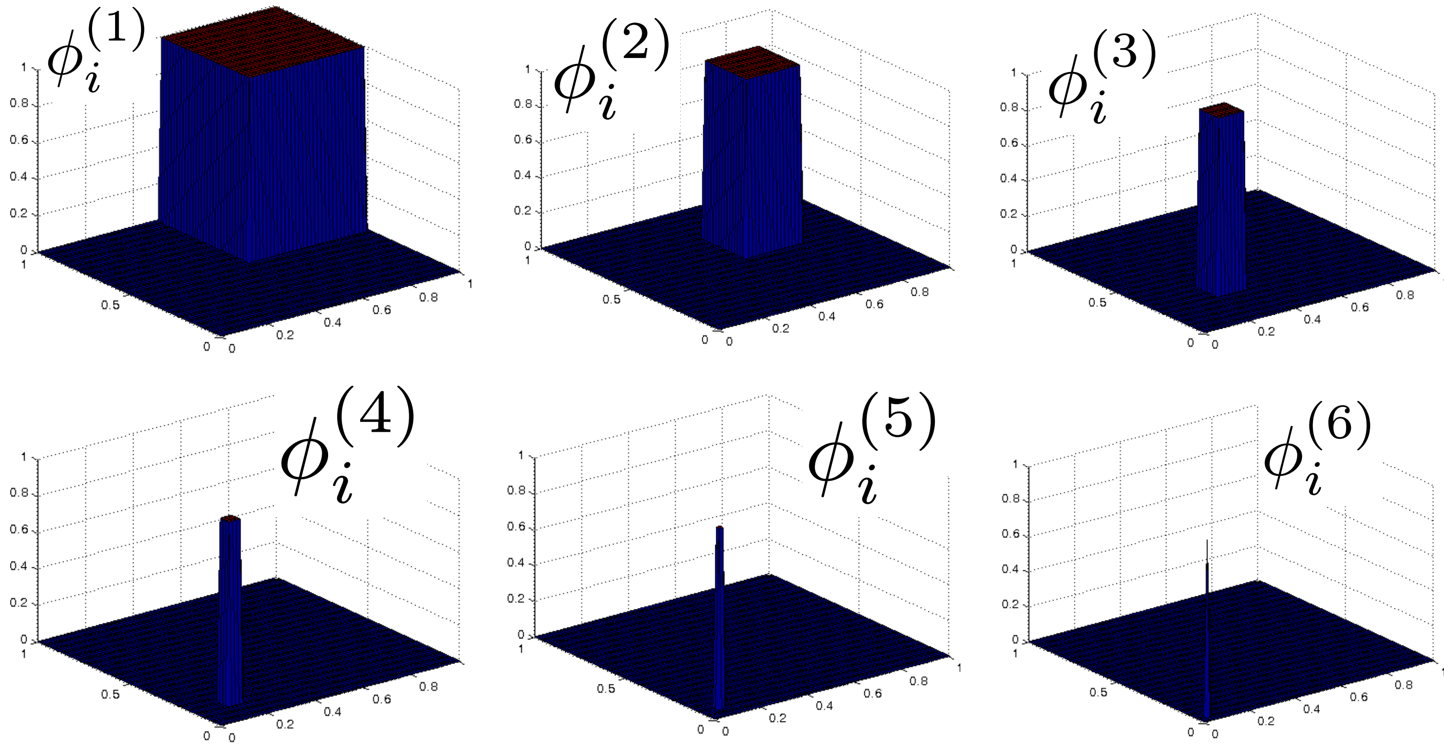
1. $|x| \leq C_s H^{-k} \|\phi\|_{H^{-s}(\Omega)}$
   for $\phi \in \{\sum_{i \in \mathcal{I}^{(k)}} x_i \phi_i^{(k)}\}$

2. $\|\phi\|_{H^{-s}(\Omega)} \leq C_s H^k |x|$
   for $\phi \in \{\sum_{i \in \mathcal{I}^{(k+1)}} x_i \phi_i^{(k+1)} \mid x \in \mathrm{Ker}(\pi^{(k,k+1)})\}$

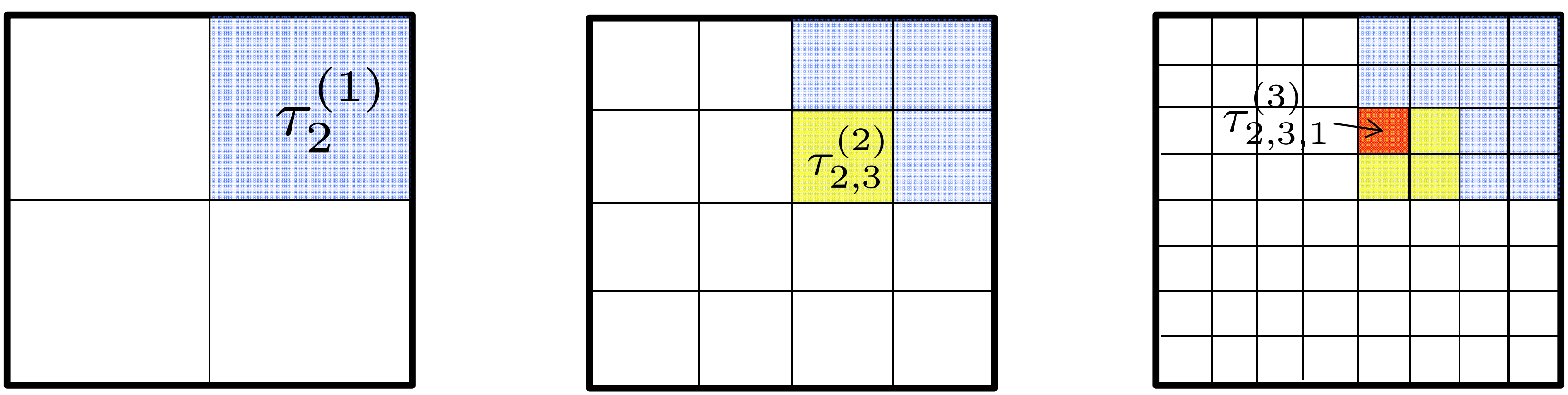
Example

$$s = 1$$

$$H = \frac{1}{2}$$



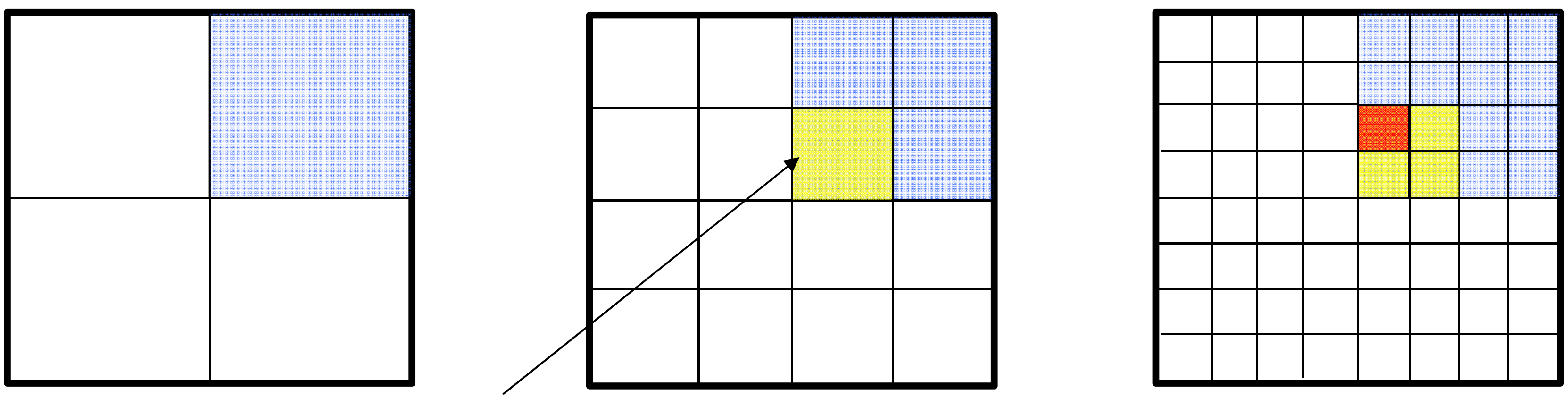
$\phi_i^{(k)}$: Weighted indicator functions of a hierarchical nested partition of Ω of resolution 2^{-k}



Example $s \geq 2$ $H = \frac{1}{2^s}$

$(\phi_{i,\alpha}^{(k)})_{\alpha \in \beth}$: orthonormal basis functions of $\mathcal{P}_{s-1}(\tau_i^{(k)})$

$\mathcal{P}_{s-1}(\tau_i^{(k)})$: polynomials of degree at most $s-1$



$\tau_i^{(k)}$: Hierarchical nested partition of Ω of resolution 2^{-k}

[H. Owhadi and C. Scovel. Universal Scalable Robust Solvers from Computational Information Games and fast eigenspace adapted Multiresolution Analysis 2017. arXiv:1703.10761]

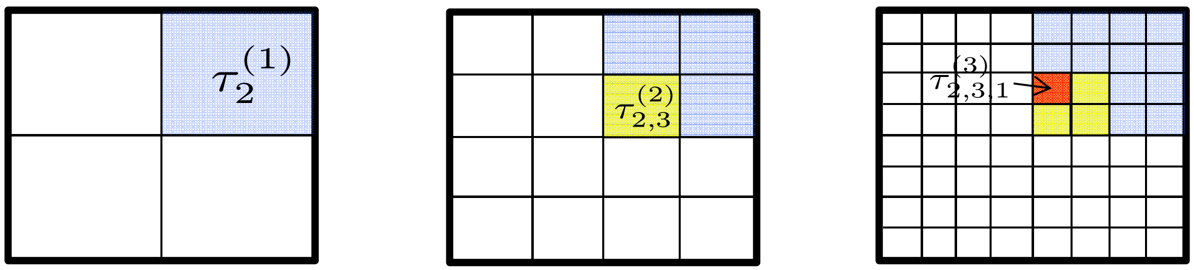
T. Y. Hou and P. Zhang. Sparse operator compression of higher order elliptic operators with rough coefficients. *To appear*, 2017.

Example

$$s \geq 2 \qquad H = \frac{1}{2^s}$$

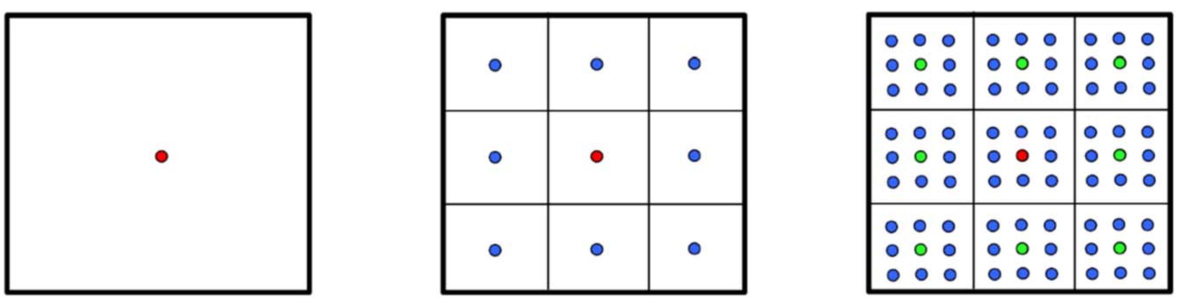
[Schäfer, Sullivan, Owhadi. 2017]: Compression, inversion, and approximate PCA of dense kernel matrices at near-linear computational complexity.

$\phi_i^{(k)}$: Weighted indicator functions of a hierarchical nested partition of Ω of resolution 2^{-k}



$$s > d/2$$

$\phi_i^{(k)}$: Subsampled delta Dirac functions



Example

$$\mathcal{B} := \mathbb{R}^N \qquad \|x\|^2 := x^T A x$$

A: $N \times N$ symmetric postive definite matrix

$$\|x\|_*^2 := x^T A^{-1} x$$

$$\|x\|_0^2 := x^T x$$

$$\phi_i^{(q)} = e_i \qquad \pi^{(k,k+1)}(\pi^{(k,k+1)})^T = I^{(k)}$$

Regularity Conditions

$$\pi^{(k,q)} = \pi^{(k,k+1)} \ldots \pi^{(q-1,q)}$$

For some $H \in (0,1)$ and $C > 0$

1. $\frac{1}{C\sqrt{\lambda_{\min}(A)}} H^k \leq \inf_{x \in \mathrm{Img}(\pi^{(q,k)})} \frac{\sqrt{x^T A^{-1} x}}{|x|}$

2. $\sup_{x \in \mathrm{Ker}(\pi^{(k,q)})} \frac{\sqrt{x^T A^{-1} x}}{|x|} \leq \frac{C}{\sqrt{\lambda_{\min}(A)}} H^k$

Conditions are covariant under quadratic form equivalence

Regularity Conditions on Primal Space

For some $H \in (0,1)$ and $C_\Phi > 0$

1. $C_\Phi^{-1} H^k \frac{\sqrt{x^T A x}}{\lambda_{\min}(A)} \leq |x|$
   for $x \in \mathrm{Img}(\pi^{(q,k)})$ and $k \in \{1, \ldots, q\}$.
2. $\inf_{y \in \mathbb{R}^{\mathcal{I}^{(k)}}} |z - \pi^{(q,k)} y| \leq C_\Phi H^k \frac{\sqrt{z^T A z}}{\lambda_{\min}(A)}$
   for $z \in \mathbb{R}^N$ and $k \in \{1, \ldots, q\}$.

Gamblet Transform/Solve

For $i \in \mathcal{I}^{(q)}$, $\psi_i^{(q)} = \varphi_i$

For $i \in \mathcal{I}^{(q)}$, $g_i^{(q)} = [g, \psi_i^{(q)}]$

For $i, j \in \mathcal{I}^{(q)}$, $A_{i,j}^{(q)} = \langle \psi_i^{(q)}, \psi_j^{(q)} \rangle$

for $k = q$ to 2 **do**

$\quad B^{(k)} = W^{(k)} A^{(k)} W^{(k),T}$

$\quad w^{(k)} = B^{(k),-1} W^{(k)} g^{(k)}$

$\quad$ For $i \in \mathcal{J}^{(k)}$, $\chi_i^{(k)} = \sum_{j \in \mathcal{I}^{(k)}} W_{i,j}^{(k)} \psi_j^{(k)}$

$\quad u^{(k)} - u^{(k-1)} = \sum_{i \in \mathcal{J}^{(k)}} w_i^{(k)} \chi_i^{(k)}$

$\quad D^{(k,k-1)} = -B^{(k),-1} W^{(k)} A^{(k)} \bar{\pi}^{(k,k-1)}$

$\quad R^{(k-1,k)} = \bar{\pi}^{(k-1,k)} + D^{(k-1,k)} W^{(k)}$

$\quad A^{(k-1)} = R^{(k-1,k)} A^{(k)} R^{(k,k-1)}$

$\quad$ For $i \in \mathcal{I}^{(k-1)}$, $\psi_i^{(k-1)} = \sum_{j \in \mathcal{I}^{(k)}} R_{i,j}^{(k-1,k)} \psi_j^{(k)}$

$\quad g^{(k-1)} = R^{(k-1,k)} g^{(k)}$

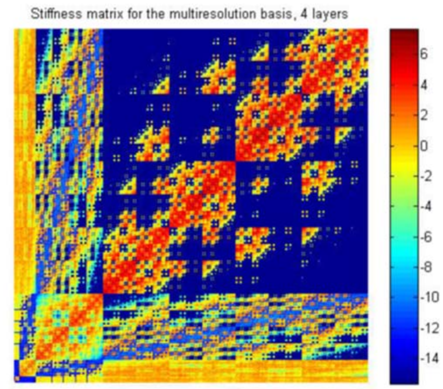
end for

$U^{(1)} = A^{(1),-1} g^{(1)}$

$u^{(1)} = \sum_{i \in \mathcal{I}^{(1)}} U_i^{(1)} \psi_i^{(1)}$

$u = u^{(1)} + (u^{(2)} - u^{(1)}) + \cdots + (u^{(q)} - u^{(q-1)})$

Fast Gamblet Transform obtained by truncation/localization



Complexity Theorem $N = \text{Card}(\mathcal{I}^{(q)})$

$N \log^{3d}(N)$: Computation of all gamblets

$N \log^{d+1}(N)$: Gamblet transform/solve of $u \in \mathcal{B}$ to accuracy H^q in $\|\cdot\|$ norm

Based on exponential decay of gamblets and locality of the operator

d: Hausdorff dimension of d^A.

d^A: Graph distance of A on $\mathcal{I}^{(q)}$

$A_{i,j} := \langle \varphi_i, \varphi_j \rangle$, stiffness matrix of the operator

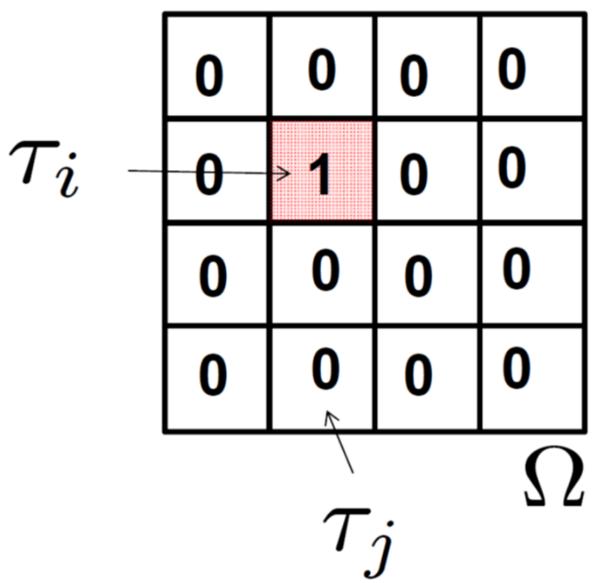
$$\text{Card}\{j | d^A_{i,j} \leq r\} \leq C\, r^d$$

Localization of Gamblets

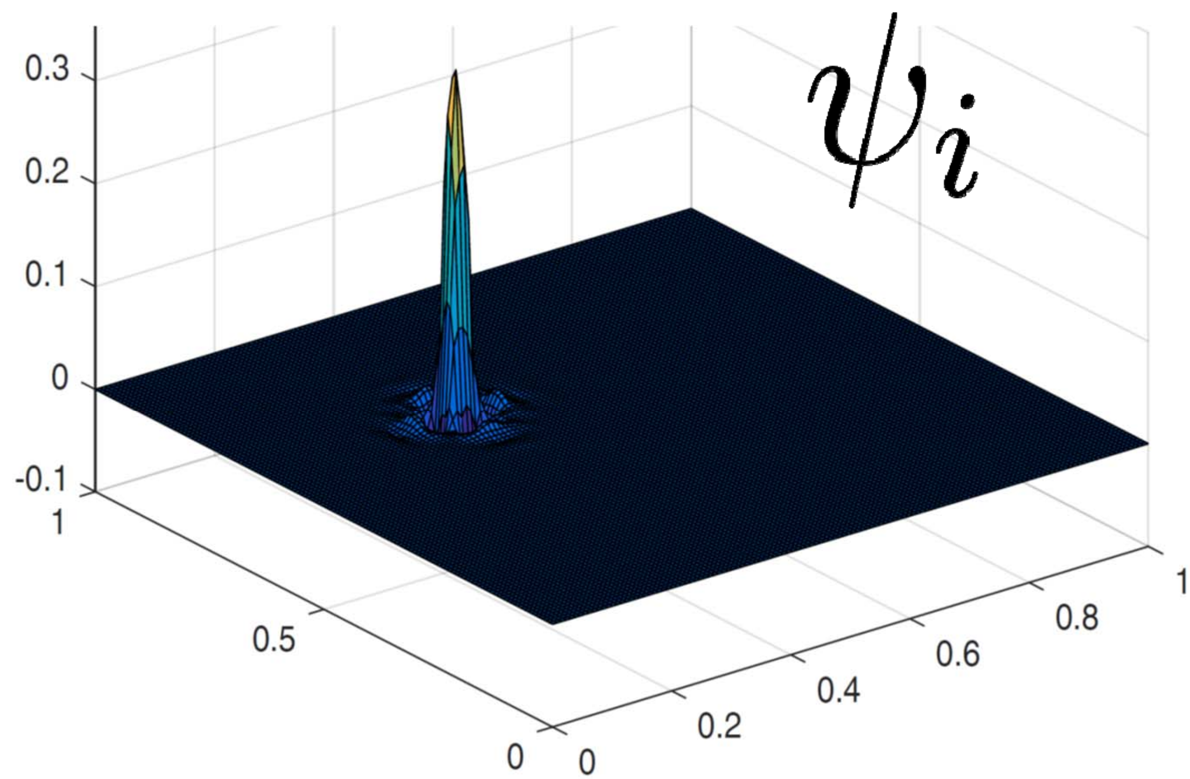
$$\mathcal{B} := H_0^s(\Omega)$$

$$\|u\|^2 := [\mathcal{L}u, u]$$

$$\psi_i = \mathbb{E}\big[\xi \mid [\phi_j, \xi] = \delta_{i,j} \text{ for } j \in \mathcal{I}\big]$$



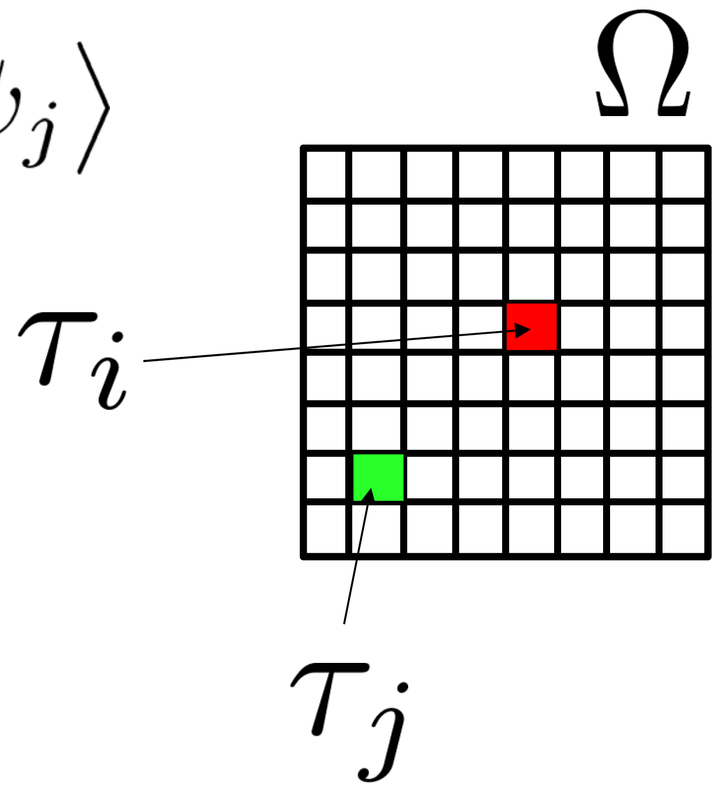
$$\phi_i = 1_{\tau_i}$$



Sparsity of the precision matrix

$$\Theta_{i,j} = \mathrm{Cov}\left([\phi_i, \xi], [\phi_j, \xi]\right)$$

$$\Theta^{-1}_{i,j} = \langle \psi_i, \psi_j \rangle$$



$$\Theta^{-1}_{i,j} = 0$$

$$\Updownarrow$$

$$\mathrm{Cov}\left([\phi_i, \xi], [\phi_j, \xi] \,\middle|\, [\phi_l, \xi], \, l \neq i, j\right) = 0$$

Localization problem in Numerical Homogenization

[Chu-Graham-Hou-2010] (limited inclusions)

[Efendiev-Galvis-Wu-2010] (limited inclusions or mask)

[Babuska-Lipton 2010] (local boundary eigenvectors)

[Owhadi-Zhang 2011] (localized transfer property)

[Malqvist-Peterseim 2012] Local Orthogonal Decomposition

[Owhadi-Zhang-Berlyand 2013] (Rough Polyharmonic Splines)

[A. Gloria, S. Neukamm, and F. Otto, 2015] (quantification of ergodicity)

[Hou and Liu,DCDS-A, 2016] [Chung-Efendiev-Hou, JCP 2016]

[Owhadi, Multiresolution operator decomposition, SIREV 2017]

[Owhadi, Zhang, gamblets for hyperbolic and parabolic PDEs, 2016]

[Hou, Qin, Zhang, 2016] [Hou, Zhang, 2017]

[Hou and Zhang, 2017]: Higher order PDEs (localization under strong ellipticity, h sufficiently small, and higher order polynomials as measurement functions)

[Kornhuber, Peterseim, Yserentant, 2016]: Subspace decomposition

Subspace decomposition/correction and Schwarz iterative methods

[J. Xu, 1992]: Iterative methods by space decomposition and subspace correction

[Griebel-Oswald, 1995]: Schwarz algorithms

Example

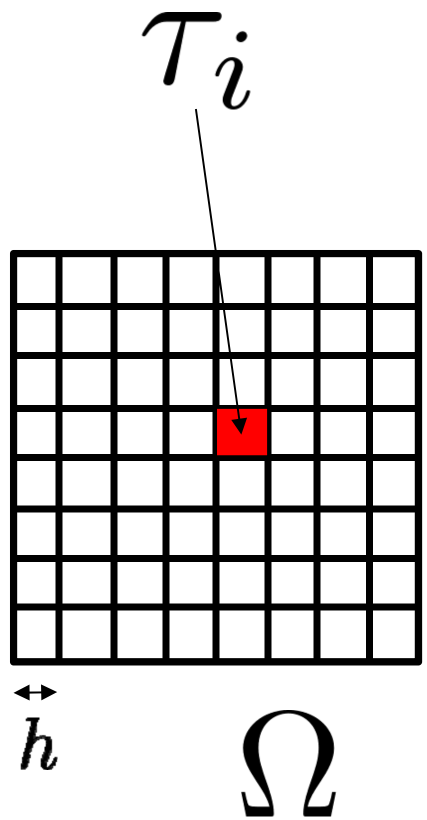
$$\mathcal{B} := H^s_0(\Omega) \qquad \|u\|^2 := [\mathcal{L}u, u]$$

$\mathcal{L}$: arbitrary continuous positive symmetric linear bijection

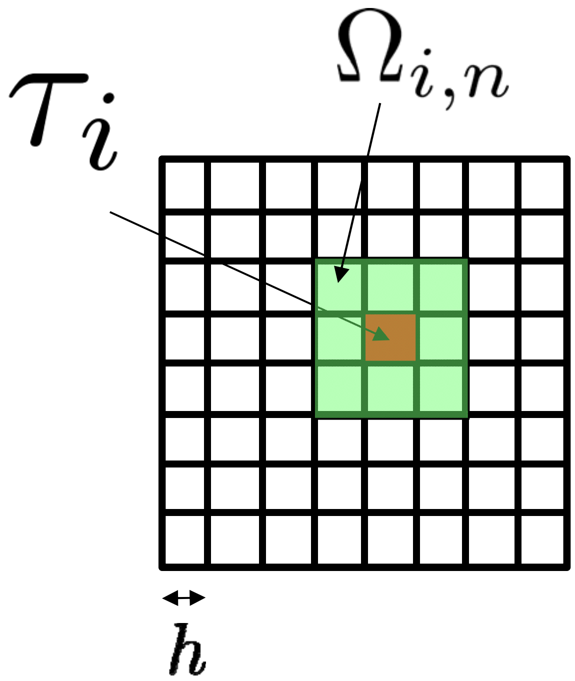
$$(H^s_0(\Omega), \|\cdot\|_{H^s_0(\Omega)}) \xrightarrow{\mathcal{L}} (H^{-s}(\Omega), \|\cdot\|_{H^{-s}(\Omega)})$$

$\mathcal{L}$ is local $\langle u, v \rangle = 0$ if u and v have disjoint supports

Examples



- $\phi_i = \frac{1_{\tau_i}}{\sqrt{|\tau_i|}}$.
- $\phi_i = \delta(\cdot - x_i)$, $(s > \frac{d}{2})$
- $(\phi_{i,\alpha})_{\alpha \in \beth}$ forms an orthonormal basis of $\mathcal{P}_{s-1}(\tau_i)$

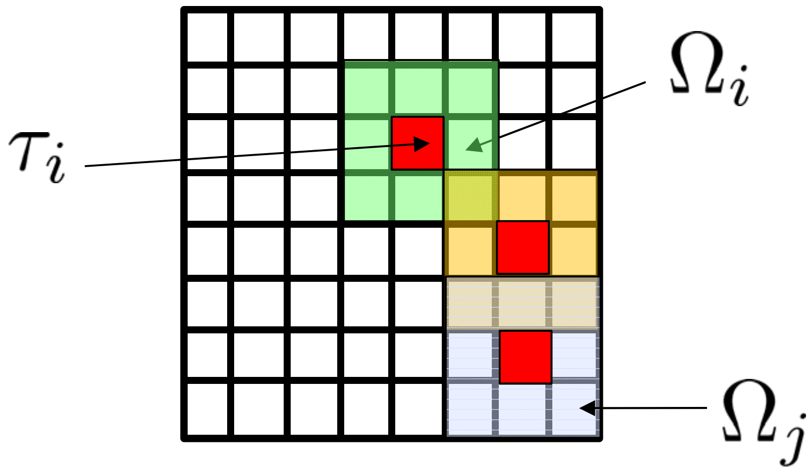


$$\mathrm{dist}(\tau_i, \partial\Omega_{i,n}) \approx nh$$

$\psi^n_{i,\alpha}$: Localization of $\psi_{i,\alpha}$ to $\Omega_{i,n}$

Theorem

$$\|\psi_{i,\alpha} - \psi^n_{i,\alpha}\|_{H^s_0(\Omega)} \leq Ce^{-n/C}$$



$$\Omega = \cup_i \Omega_i$$

$$H_0^s(\Omega) = \sum_{i \in \beth} H_0^s(\Omega_i)$$

Condition for localization

For $\varphi \in H^{-s}(\Omega)$

$$C_{\min} \le \frac{\sum_i \inf_{\phi \in \Phi} \|\varphi - \phi\|^2_{H^{-s}(\Omega_i)}}{\inf_{\phi \in \Phi} \|\varphi - \phi\|^2_{H^{-s}(\Omega)}} \le C_{\max}$$

$$\Phi = \{\phi_{i,\alpha} \mid (i,\alpha) \in \beth \times \aleph\}$$

Theorem

Assume that there exists a constant C_0 such that $|\aleph| \le C_0$,

- $\|D^t f\|_{L^2(\Omega)} \le C_0 h^{s-t} \|f\|_{H_0^s(\Omega)}$ for $t \in \{0, 1, \ldots, s\}$, for $f \in H_0^s(\Omega)$ such that $[\phi_{i,\alpha}, f] = 0$ for $(i, \alpha) \in \beth \times \aleph$,
- $\sum_{i \in \beth, \alpha \in \aleph} [\phi_{i,\alpha}, f]^2 \le C_0 \big(\|f\|^2_{L^2(\Omega)} + h^{2s} \|f\|^2_{H_0^s(\Omega)} \big)$, for $f \in H_0^s(\Omega)$, and
- $|x|^2 \le C_0 h^{-2s} \| \sum_{\alpha \in \aleph} x_\alpha \phi_{i,\alpha} \|^2_{H^{-s}(\tau_i)}$, for $i \in \beth$ and $x \in \mathbb{R}^{\aleph}$.

Then for $\varphi \in H^{-s}(\Omega)$

$$C_{\min} \le \frac{\sum_i \inf_{\phi \in \Phi} \|\varphi - \phi\|^2_{H^{-s}(\Omega_i)}}{\inf_{\phi \in \Phi} \|\varphi - \phi\|^2_{H^{-s}(\Omega)}} \le C_{\max}$$

Where $C_{\max}, C_{\min}$ depend only on C_0, d, δ and s

Banach space setting

$$\mathcal{B} = \sum_{i \in \beth} \mathcal{B}_i$$

$\| \cdot \|_i$ and $\| \cdot \|_{i,*}$ norms induced by $\| \cdot \|$ on $\mathcal{B}_i$ and $\mathcal{B}_i^*$

Condition for localization

For $\varphi \in \mathcal{B}^*$

$$C_{\min} \leq \frac{\sum_i \inf_{\phi \in \Phi} \|\varphi - \phi\|_{i,*}^2}{\inf_{\phi \in \Phi} \|\varphi - \phi\|_*^2} \leq C_{\max}$$

$$\Phi = \{\phi_{i,\alpha} \mid (i, \alpha) \in \beth \times \aleph\}$$

Operator connectivity distance

C: $\beth \times \beth$ connectivity matrix

$C_{i,j} = 1$ if $\exists(\chi_i, \chi_j) \in \mathcal{B}_i \times \mathcal{B}_j$ s.t. $\langle \chi_i, \chi_j \rangle \neq 0$

$C_{i,j} = 0$ otherwise

$\mathbf{d}$: Graph distance on $\beth$ induced by C

$\psi^n_{i,\alpha}$: Localization of $\psi_{i,\alpha}$ to $\mathcal{B}^n_i$

$$\mathcal{B}^n_i = \cup_{j:\mathbf{d}(i,j)\leq n} B_i$$

Theorem Under localization conditions

$$\|\psi_{i,\alpha} - \psi^n_{i,\alpha}\| \leq Ce^{-n/C}$$

Thank you

- **Universal Scalable Robust Solvers from Computational Information Games and fast eigenspace adapted Multiresolution Analysis, 2017. arXiv:1703.10761. H. Owhadi and C. Scovel.**
- Compression, inversion, and approximate PCA of dense kernel matrices at near-linear computational complexity, Schäfer, Sullivan, Owhadi. 2017.
- Multigrid with gamblets. L. Zhang and H. Owhadi, 2017
- Gamblets for opening the complexity-bottleneck of implicit schemes for hyperbolic and parabolic ODEs/PDEs with rough coefficients, 2016. H. Owhadi and L. Zhang. arXiv:1606.07686
- Multigrid with rough coefficients and Multiresolution operator decomposition from Hierarchical Information Games. H. Owhadi. SIAM Review, 59(1), 99149, 2017. arXiv:1503.03467
- Towards Machine Wald (book chapter). Houman Owhadi and Clint Scovel. Springer Handbook of Uncertainty Quantification, 2016, arXiv:1508.02449.
- Bayesian Numerical Homogenization. H. Owhadi. SIAM Multiscale Modeling & Simulation, 13(3), 812828, 2015. arXiv:1406.6668


Florian Schäfer


Clint Scovel


Tim Sullivan


Lei Zhang

DARPA EQUiPS / AFOSR award no FA9550-16-1-0054
(Computational Information Games)