Super-Diffusivity in a Shear Flow Model from Perpetual Homogenization

Gérard Ben Arous[1]**, Houman Owhadi**[2]

[1] DMA, EPFL, 1015 Lausanne, Switzerland. E-mail: gerard.benarous@epfl.ch
[2] William Davidson Faculty (Bloomfield), Technion, 32000 Haifa, Israel.
E-mail: owhadi@technix.technion.ac.il

Abstract: This paper is concerned with the asymptotic behavior solutions of stochastic differential equations $dy_t = d\omega_t - \nabla\Gamma(y_t)dt$, $y_0 = 0$ and $d = 2$. Γ is a 2×2 skew-symmetric matrix associated to a shear flow characterized by an infinite number of spatial scales $\Gamma_{12} = -\Gamma_{21} = h(x_1)$, with $h(x_1) = \sum_{n=0}^{\infty} \gamma_n h^n(x_1/R_n)$, where h^n are smooth functions of period 1, $h^n(0) = 0$, γ_n and R_n grow exponentially fast with n. We can show that y_t has an anomalous fast behavior ($\mathbb{E}[|y_t|^2] \sim t^{1+\nu}$ with $\nu > 0$) and obtain quantitative estimates on the anomaly using and developing the tools of homogenization.

Contents

1. Introduction

Turbulent incompressible flows are characterized by multiple scales of mixing length and convection rolls. It is heuristically known and expected that a diffusive transport in such media will be super-diffusive. The first known observation of this anomaly is attributed to Richardson [27] who analyzed available experimental data on diffusion in air, varying on about 12 orders of magnitude. On that basis, he empirically conjectured that the diffusion coefficient D_λ in turbulent air depends on the scale length λ of the measurement. The Richardson law,

$$D_\lambda \propto \lambda^{\frac{4}{3}} \tag{1}$$

was related to Kolmogorov–Obukhov turbulence spectrum, $v \propto \lambda^{\frac{1}{3}}$, by Batchelor [4]. The super-diffusive law of the root-mean-square relative displacement $\lambda(t)$ of advected particles

$$\lambda(t) \propto (D_{\lambda(t)}t)^{\frac{1}{2}} \propto t^{\frac{3}{2}} \tag{2}$$

was derived by Obukhov [22] from a dimensional analysis similar to the one that led Kolmogorov [18] to the $\lambda^{\frac{1}{3}}$ velocity spectrum.

More recently physicists and mathematicians have started to investigate the super-diffusive phenomenon (from both heuristic and rigorous points of view) by using the tools of homogenization or renormalization; we refer to M. Avellaneda and A. Majda [3, 2], J. Glimm et al. [12, 13, 15], J. Glimm and Q. Zhang [16], Q. Zhang [28], M.B. Isichenko and J. Kalda [17], G. Gaudron [14].

It is now well known that homogenization over a periodic or ergodic divergence free drift has the property to enhance the diffusion [10, 11, 19]. It is also expected that several spatial scales of eddies should give rise to anomalous diffusion between proper time scales outside the homogenization regime or when the bigger scale has not yet been homogenized. We refer to M. Avellaneda [1]; A. Fannjiang [8]; Rabi Bhattacharya [6] (see also [7] by Bhattacharya, Denker and Goswami); A. Fannjiang and T. Komorowski [9], and this panorama is certainly not complete.

The purpose of this paper is to implement rigorously on a shear flow model the idea that the key to anomalous fast diffusion in turbulent flows is an unfinished homogenization process over a large number of scales of eddies without a sharp separation between them. We will assume that the ratios between the spatial scales are bounded. The underlying phenomenon is similar to the one related to anomalous slow diffusion from perpetual homogenization on an infinite number of scales of gradient drifts [25, 5], the main difference lies in the asymptotic behavior of the multi-scale effective diffusivities $D(n)$ associated with n spatial scales, i.e. $D(n)$ diverge towards ∞ or converge towards 0 with exponential rate depending on the nature of the scales: eddies or obstacles.

Note that the shear-layer model is exactly solvable ([3, 14]). When the geometrically divergent scales are recast into the Fourier setting with a power-law spectrum, super-diffusivity has already been proven in the limit $t \to \infty$. Our purpose in this paper is to show that never-ending homogenization can be used as a tool to obtain a quantitative control on the anomaly for finite times, not just an asymptotic result and without any self-similarity assumption. Moreover it will be shown that the mean-squared displacement $\mathbb{E}[y_t^2]$ of the diffusion in the shear flow behaves like $D(n(t))t$ (see (24)). In this formula, $n(t)$ has a logarithmic growth and corresponds to the number of scales that can be considered as homogenized at time t, casting into light the role of never-ending homogenization in the anomalous fast behavior of a diffusion process in a shear-flow model. Moreover it will be shown in [26] that the strategy associated to never-ending homogenization can be extended to higher dimensions (and non shear flow models of turbulence). We would like to refer the reader to an interesting and related recent preprint by S. Olla and T. Komorowski [29] on "the superdiffusive behavior of passive tracer with a Gaussian drift".

2. The Model

Let us consider in dimension two a Brownian motion with a drift given by the divergence of a shear flow stream matrix, i.e. the solution of the stochastic differential equation:

$$dy_t = d\omega_t - \nabla \Gamma(y_t) dt, \quad y_0 = 0, \tag{3}$$

where Γ is a skew-symmetric 2×2 shear flow matrix,

$$\Gamma(x_1, x_2) = \begin{pmatrix} 0 & h(x_1) \\ -h(x_1) & 0 \end{pmatrix}. \tag{4}$$

The function $(x_1, x_2) \rightarrow h(x_1)$ is given by a sum of infinitely many periodic functions with (geometrically) increasing periods

$$h(x_1) = \sum_{n=0}^{\infty} \gamma_n h_n \left(\frac{x_1}{R_n} \right), \tag{5}$$

where h_n are smooth functions of period 1. We will assume that

$$h_n(0) = 0. \tag{6}$$

We will normalize the functions h_n by the choosing their variance equal to one:

$$\mathrm{Var}(h_n) = \int_0^1 (h_n(x) - \int_0^1 h_n(y) dy)^2 dx = 1. \tag{7}$$

R_n and γ_n grow exponentially fast with n, i.e.

$$R_n = \prod_{k=0}^{n} r_k, \tag{8}$$

where r_n are integers, $r_0 = 1$,

$$\rho_{\min} = \inf_{n \in \mathbb{N}^*} r_n \geq 2 \quad \text{and} \quad \rho_{\max} = \sup_{n \in \mathbb{N}^*} r_n < \infty. \tag{9}$$

We choose $\gamma_0 = 1$ and

$$\gamma_{\min} = \inf_{n \in \mathbb{N}} (\gamma_{n+1}/\gamma_n) > 1 \quad \text{and} \quad \gamma_{\max} = \sup_{n \in \mathbb{N}} (\gamma_{n+1}/\gamma_n) < \infty. \tag{10}$$

It is assumed that the first derivate of the potentials h_n are uniformly bounded. ($\mathrm{Osc}(h)$ stands for $\sup h - \inf h$)

$$K_0 = \sup_{n \in \mathbb{N}} \mathrm{Osc}(h_n) < \infty, \qquad K_1 = \sup_{n \in \mathbb{N}} \|h_n'\|_\infty < \infty. \tag{11}$$

In this paper we shall distinguish two hypotheses

Hypothesis 1.

$$\rho_{\min} > \gamma_{\max}. \tag{12}$$

Hypothesis 2.

$$\rho_{\min} > \gamma_{\max}^{1/2}. \tag{13}$$

For all $n \in \mathbb{N}$,

$$h'_n(0) = 0 \tag{14}$$

and

$$K_2 = \sup_{n \in \mathbb{N}} \|\partial_1^2 h_n\|_\infty < \infty. \tag{15}$$

Let us observe that under Hypotheses 1, (6), (9), (10) and (11) h is a well defined C^1 function on $\mathbb{R}^d$ and

$$|h(x)| \le K_1 |x|(1 - \gamma_{\max}/\rho_{\min})^{-1} \quad |h'(x)| \le K_1(1 - \gamma_{\max}/\rho_{\min})^{-1}. \tag{16}$$

Thus under Hypothesis 1, Γ is a well defined Lipschitz stream matrix and the solution of the stochastic differential equation (3) exists; is unique up to sets of measure 0 with respect to the Wiener measure and is a strong Markov continuous Feller process.

Under Hypothesis 2, (6), (9), (10) and (11), Γ is no more Lipschitz continuous but h is still a well defined C^2 function on $\mathbb{R}^d$ and

$$|h(x)| \le K_2 |x|^2 (1 - \gamma_{\max}/\rho_{\min}^2)^{-1} \quad |h'(x)| \le K_2(1 - \gamma_{\max}/\rho_{\min}^2)^{-1}|x|. \tag{17}$$

3. Main Results

3.1. Under Hypothesis 1. Our objective is to show that the solution (3) is abnormally fast and the asymptotic sub-diffusivity will be characterized as an anomalous behavior of the variance at time t, i.e. $\mathbb{E}_0[y_t^2] \sim t^{1+\nu}$ as $t \to \infty$. More precisely there exists a constant $\rho_0(\gamma_{min}, \gamma_{max}, K_0, K_1)$ and a time $t_0(\gamma_{min}, \gamma_{max}, R_1, K_0, K_1)$ such that

Theorem 1. *If $\rho_{min} > \rho_0$ and y_t is a solution of (3) then for $t > t_0$,*

$$\mathbb{E}_0[|y_t.e_2|^2] = t^{1+\nu(t)} \tag{18}$$

with

$$\frac{\ln \gamma_{\min}}{\ln \rho_{max} + \ln \frac{\gamma_{\max}}{\gamma_{\min}}} - \frac{C_1}{\ln t} \le \nu(t) \le \frac{\ln \gamma_{\max}}{\ln \rho_{min} + \ln \frac{\gamma_{\min}}{\gamma_{\max}}} + \frac{C_2}{\ln t}, \tag{19}$$

where the constants C_1 and C_2 depend on $\rho_{\min}, \gamma_{\min}, \gamma_{\max}, \rho_{\max}, K_0, K_1$.

We remark that if $\gamma_{\max} = \gamma_{\min} = \gamma$ and $\rho_{\max} = \rho_{\min} = \rho$ then $\nu(t) \sim \ln \gamma / \ln \rho$.
The key of the fast asymptotic behavior of the variance of the solution of (3) is the geometric rate of divergence towards ∞ of the multi-scale effective matrices associated to a finite number of scales. More precisely, for $k, p \in \mathbb{N}$, $k \le p$ we will write

$$H^{k,p} = \sum_{n=k}^{p} \gamma_n h_n(x/R_n) \tag{20}$$

and $\Gamma^{k,p}$ the skew-symmetric matrix given by $\Gamma_{1,2}^{k,p}(x_1, x_2) = H^{k,p}(x_1)$. Let $D(\Gamma^{0,p})$ be the effective diffusivity associated to homogenization of the periodic operator $L_{\Gamma^{0,p}} = 1/2\Delta - \nabla\Gamma^{0,p}\nabla$. Then it is easy to see that

$$D(\Gamma^{0,p}) = \begin{pmatrix} 1 & 0 \\ 0 & D(\Gamma^{0,p})_{22} \end{pmatrix} \tag{21}$$

and it will be shown that

Theorem 2. *For*

$$\epsilon = 4K_1\big(\rho_{\min}(\gamma_{\min} - 1)\big)^{-1} < 1, \tag{22}$$

$$1 + 4(1 - \epsilon)\sum_{k=0}^{p}\gamma_k^2 \le D(\Gamma^{0,p})_{22} \le 1 + 4(1 + \epsilon)\sum_{k=0}^{p}\gamma_k^2. \tag{23}$$

The super-diffusive behavior can be explained and controlled by a perpetual homogenization process taking place over the infinite number of scales $0, \dots, n, \dots$. The idea of the proof of Theorem 1 is to distinguish, when one tries to estimate (18), the smaller scales which have already been homogenized $(0, \dots, n_{ef}$ called effective scales), the bigger scales which have not had a visible influence on the diffusion $(n_{dri}, \dots, \infty$ called drift scales because they will be replaced by a constant drift in the proof) and some intermediate scales that manifest their particular shapes in the behavior of the diffusion $(n_{ef} + 1, \dots, n_{dri} - 1 = n_{ef} + n_{per}$ called perturbation scales because they will enter in the proof as a perturbation of the homogenization process over the smaller scales).

The number of effective scales is fixed by the mean squared displacement of $y_t.e_1$. Writing $n_{ef}(t) = \inf\{n : t \le R_n^2\}$ one proves that

$$\mathbb{E}[(y_t.e_2)^2] \sim D(\Gamma^{0,n_{ef}(t)})t. \tag{24}$$

Assume for instance $R_n = \rho^n$ and $\gamma_n = \gamma^n$ then $n_{ef}(t) \sim \ln t/(2\ln\rho)$ and

$$\mathbb{E}[(y_t.e_2)^2] \sim t^{1 + \frac{\ln\gamma}{\ln\rho}}.$$

We remark that the quantitative control is sharper than the one associated to a perpetual homogenization on a gradient drift [25]; this is explained by the fact that the number of perturbation scales is limited to only one scale with a divergence free drift. Nevertheless the main difficulty is to control the influence of this intermediate scale and the core of that control is based on the following mixing stochastic inequality (we write $\mathbb{T}_R^d = R\mathbb{T}^d$ the torus of dimension d and side R).

Proposition 1. *Let* $R > 0$ *and* $f, G \in (H^1(\mathbb{T}_R^1))^2$ *such that* $\int_0^1 f(y)dy = 0$ *and* $\int_0^1 G(y)dy = 0$, *let* $r > 0$ *and* $t > 0$,

$$\left|\mathbb{E}\Big[G(b_t)\int_0^t \partial_1 f(rb_s)\,ds\Big]\right| \le \|f\|_{L^2(\mathbb{T}_R^1)}\|G'\|_{L^2(\mathbb{T}_R^1)}2r^{-1}. \tag{25}$$

3.2. Physical interpretation. We want to emphasize that to obtain a super-diffusive be-
havior of the solution of (3) of the shape $\mathbb{E}[y_t^2] \sim t^{1+\nu}$ with $\nu > 0$ it is necessary to
assume the exponential rate of growth of the parameters γ_n; this has a clear meaning
when the flow is compared on a heuristic point of view to a real turbulent flow.

First, note that our model starts with the dissipation scale and expresses the inertial
range of scales as geometrically divergent. Next, the parameters $\gamma_n \|h_n'\|_\infty / R_n$ represents
the amplitude of the pulsations of the eddies of size R_n. It is a well known characteristic
of turbulence ([20] p. 129) that the amplitude of the pulsations increase with the scale,
since for all scales $\|h_n'\|_\infty \le K_1$ one should have $\lim \gamma_n / R_n \to \infty$ to reflect that image.
In our model, it is sufficient to assume the exponential rate of divergence of γ_n to obtain
a super-diffusive behavior.

Let us also notice on a heuristic point of view (our model is not isotropic and does
not depend on the time) that the energy dissipated per unit time and unit volume in the
eddies of scale n is of order of

$$\epsilon_n \propto \frac{\gamma_n^2}{R_n^4} K_2^2. \tag{26}$$

So saying that the energy is dissipated mainly in the small eddies is equivalent to saying
that $\gamma_n / R_n^2 \to 0$ as $n \to \infty$ or if $R_n = \rho^n$ and $\gamma_n = \rho^{\alpha n}$, this equivalent to say that
$\alpha < 2$, which is included in Hypothesis 2.

The Kolmogorov–Obukhov law is equivalent to say that $K_2 < \infty$, for all n $h_n'(0) = 0$
and

$$\gamma_n \propto R_n^{\frac{4}{3}}, \tag{27}$$

or if $R_n = \rho^n$ and $\gamma_n = \rho^{\alpha n}$, this is equivalent to say that $\alpha = \frac{4}{3}$ corresponding to the
Kolmogorov spectrum, which violates the hypothesis 1, $\rho > \gamma$ but not Hypothesis 2
($\rho > \gamma^{\frac{1}{2}}$) which will be addressed below.

Overlapping ratios. The super-diffusive behavior in Theorem 1 requires a minimal
separation between scales, i.e. $\rho_{\min} > \rho_0$ and this condition is necessary. Assume for
instance $R_n = \rho^n$ and $\gamma_n = \gamma^n$, then if $h_n(x_1) = g(x_1) - \gamma^p g(a^p x_1)$ (with $g \in C^1(\mathbb{T}^1)$
where $\mathbb{T}^1$ is the torus $\mathbb{R}/\mathbb{Z}$ and $a \in \mathbb{N}^*$) it is easy to see that for $\rho = a$, Γ is bounded and
y_t has a normal behavior by Norris's Aronson type estimates [21]. Thus, as for a gradient
drift [25], it is easy to see for simple examples that when $\rho \le \rho_0$ the solution of (3) may
have a normal or a super-diffusive behavior depending on the value of ρ and the shapes
of h_n and ratios of normal behavior may be surrounded by ratios of anomalous behavior.
This phenomenon is created by a strong overlap between spatial ratios between scales.

Thus the hypothesis (12) is necessary not only to ensure that one has a well defined
C^1 stream matrix Γ but also that its associated diffusion may show an anomalous fast
behavior. Indeed with $h_n(x) = \sin(2\pi x) - 3\sin(6\pi x)$, $\gamma_n = \rho_n = 3^n$ one has $\|h\|_\infty <
\infty$, which leads to a normal behavior of y_t.

The case $\rho_{\min} \le \gamma_{\max}$ will be addressed with Hypothesis 2. Let us observe that to
investigate this case we had to add further information on the higher derivate of h_n' to
ensure that h is well defined: $h_n'(0) = 0$, $\|h_n''\|_\infty \le K_2$ under the constraint $\rho_{\min} > \gamma_{\max}^2$
(which includes the Kolmogorov case $\gamma = \rho^{4/3}$).

3.3. Under Hypothesis 2. In this subsection we will give theorems putting into evidence the anomalous fast behavior of the solutions of (3) under Hypothesis 2 which includes the Kolmogorov spectrum.

Theorem 3. *Under Hypothesis (2), if there exists a constant $z > 2$ such that $\gamma_{\min} \geq z$ then there exists a constant C_0 depending on z, K_0, K_2 such that if*

$$\rho_{\min}^2 \gamma_{\max}^{-1} > C_0 \tag{28}$$

then there exist constants $C_1 > 0$ depending on z, $C_2 > 0$ depending on z, K_0, K_2, $\gamma_{\max}$, $C_3 > 0$ on z, K_0 and $C_4 > 0$ on z, K_0, $\rho_{\max}$ such that if y_t is a solution of (3) and $t \geq C_3$, then

$$C_1 t \gamma_{p(t)}^2 \leq \mathbb{E}\big[(y_t.e_2)^2\big] \leq C_2 t \gamma_{p(t)}^2 \tag{29}$$

and

$$C_4 t^{1+v(t)} \leq \mathbb{E}\big[(y_t.e_2)^2\big] \leq C_2 t^{1+v(t)} \tag{30}$$

with

$$p(t) := \sup\{n \in \mathbb{N} : 16(1 + K_0^2)\big(1 - (\gamma_{\min} - 1)^{-1}\big)^{-2} R_n^2 < t\} \tag{31}$$

and

$$v(t) := \ln \gamma_{p(t)} \big/ \ln R_{p(t)}. \tag{32}$$

Remark. It is important to note that Eq. (29) emphasizes the role of never-ending homogenization in the fast behavior of the diffusion and its proof is also based on the separation of the infinite number of scales into smaller ones which act through their effective properties, larger ones which will be bounded by a constant drift and an intermediate one that one has to control. Equation (30) gives quantitative control of the anomaly without any self-similarity assumption; $v(t)$ is not a constant if the multi-scale velocity distribution associated to (3) is not self-similar but one has

$$\ln \gamma_{\min} \big/ \ln \rho_{\max} \leq v(t) \leq \ln \gamma_{\max} \big/ \ln \rho_{\min}. \tag{33}$$

Definition 1. *The Stochastic Differential Equation (3) is said to have a self-similar velocity distribution if and only if*

$$\rho_{\min} = \rho_{\max} = \rho \tag{34}$$

and

$$\gamma_{\min} = \gamma_{\max} = \gamma. \tag{35}$$

Then we will write

$$\alpha = \ln \gamma / \ln \rho. \tag{36}$$

From Theorem 3 one easily deduces the following theorem.

Theorem 4. *Assume that the SDE (3) has a self-similar velocity distribution with $0 < \alpha < 2$. Under Hypothesis (2), there exists a constant C_0 depending on α, K_0, K_2 such that if*

$$\rho > C_0 \tag{37}$$

then there exist constants $C_1 > 0$ depending on K_0, ρ, $C_2 > 0$ on K_0, K_2, ρ, α and C_3 on K_0 such that if y_t is a solution of (3) and $t \geq C_3$, then

$$C_1 t^{1+\alpha} \leq \mathbb{E}[(y_t.e_2)^2] \leq C_2 t^{1+\alpha}. \tag{38}$$

Remark. Observe that all the velocity distribution $0 < \alpha < 2$ is covered by Theorem 4. The condition (37) is still needed to avoid overlapping ratios and cocycles.

It is important to note that even if Theorem 3 allows to cover a larger spectrum of velocity distribution than Theorem 1, its proof is based on the regularity of the drift associated to the diffusion ($K_2 < \infty$). Whereas in Theorem 1, the drift $-\nabla \Gamma$ associated to the stochastic differential equation (3) can be non-Holder-continuous (only the regularity of Γ is needed to prove an anomalous fast behavior).

3.4. Remark. Fast separation between scales. When $\gamma_{min} > 1$ and $\gamma_{max} < \infty$, the feature that distinguishes a strong anomalous behavior from a weak one is the rate at which spatial scales do separate. Indeed one can follow the proof of Theorem 1, changing the condition $\rho_{max} < \infty$ into $R_n = R_{n-1}[\rho^{n^\alpha}/R_{n-1}]$ ($\rho, \alpha > 1$) and $\gamma_{max} = \gamma_{min} = \gamma$ to obtain

Theorem 5. *For $t > t_0(\gamma_2, R_2, K_0, K_1)$,*

$$C_1 t\gamma^{\beta(t)} \leq \mathbb{E}_0[|y_t.e_2|^2] \leq C_2 t\gamma^{\beta(t)}, \tag{39}$$

$$with \quad \beta(t) = 2(2\ln \rho)^{-\frac{1}{\alpha}}(\ln t)^{\frac{1}{\alpha}}, \tag{40}$$

where the constants C_1 and C_2 depend on ρ, γ, α, K_1.

And as $\alpha \downarrow 1$ the behavior of the solution of (3) pass from weakly anomalous to strongly anomalous.

4. Proofs Under Hypothesis 1

4.1. Notations and proof of Theorem 2. In this subsection we will prove Theorem 2 using the explicit formula of the effective diffusivity. We will first introduce the basic notations that will also be used to prove Theorem 1. Let J be smooth T_1^2 periodic 2×2 skew-symmetric matrix such that $J_{12}(x_1, x_2) = j(x_1)$ and consider the periodic operator $L_J = 1/2\Delta - \nabla J\nabla$. We call χ_l the solution of the cell problem associated to L_J, i.e. the T_1^2 periodic solution of $L_J(\chi_l - l.x) = 0$ with $\chi_l(0) = 0$. One easily obtains that

$$\chi_l(x_1, x_2) = -2l_2\left[\int_0^{x_1} j(y)dy - x_1\int_0^1 j(y)dy\right]. \tag{41}$$

The solution of the cell problem allows to compute the effective diffusivity ${}^t l D(J) l = \int_{T_1^2} |l - \nabla \chi_l(x)|^2 dx$ that leads to

$$D(J) = \begin{pmatrix} 1 & 0 \\ 0 & 1 + 4\,\text{Var}(j) \end{pmatrix}. \tag{42}$$

For a R periodic function f we will write

$$\text{Var}(f) = 1/R \int_0^R (f(x) - \int_0^R f(y)dy)^2 dx. \tag{43}$$

Using the notation (20), from Eq. (42) we obtain (21) with

$$D(\Gamma^{0,p})_{2,2} = 1 + 4\,\text{Var}(H^p). \tag{44}$$

Now we will prove by induction on p that (for ϵ given by (22))

$$(1 - \epsilon) \sum_{k=0}^p \gamma_k^2 \leq \text{Var}(H^p) \leq (1 + \epsilon) \sum_{k=0}^p \gamma_k^2. \tag{45}$$

Then Eq. (23) of theorem will follow by (44). Equation (45) is trivially true for $p = 0$.
From the explicit formula of H^p we will show in paragraph 4.1 that

$$\left| \text{Var}(H^p) - \text{Var}(H^{p-1}) - \gamma_p^2 \right| \leq 2\gamma_p K_1 r_p^{-1} \sqrt{\text{Var}(H^{p-1})}. \tag{46}$$

Assuming (45) is true at the rank p one obtains

$$\sqrt{\text{Var}(H^p)} \leq (1 + \epsilon)^{\frac{1}{2}} \gamma_{p+1} \left(\sum_{k=0}^p (\gamma_k/\gamma_{p+1})^2 \right)^{\frac{1}{2}}$$

$$\leq (1 + \epsilon)^{\frac{1}{2}} \gamma_{p+1} (\gamma_{\min} - 1)^{-1}, \tag{47}$$

and it is easy to see that the condition (22) implies that $\epsilon \geq 2K_1(1 + \epsilon)^{\frac{1}{2}}(\rho_{\min}(\gamma_{\min} - 1))^{-1}$; combining this with (46) and (47) one obtains that

$$\left| \text{Var}(H^{p+1}) - \text{Var}(H^p) - \gamma_{p+1}^2 \right| \leq \epsilon \gamma_{p+1}^2 \tag{48}$$

which proves the induction and henceforth the theorem.

4.1.1. From the equation

$$\text{Var}(H^p) = \int_0^1 \left((H^{p-1}(R_p x) - \int_0^1 H^{p-1}(R_p y)dy) + \gamma_p (h_p(x) - \int_0^1 h_p(y)dy) \right)^2 dx$$

one obtains

$$\left| \text{Var}(H^p) - \text{Var}(H^{p-1}) - \gamma_p^2 \right| \leq 2\gamma_p |E| \tag{49}$$

with

$$E = \text{Cov}(S_{R_p} H^{p-1}, h_p),$$

where Cov stands for the covariance: $\text{Cov}(f, g) = \int_0^1 (f(x) - \int_0^1 f(y)dy)(g(x) - \int_0^1 g(y)dy)dx$ and S_R is the scaling operator $S_R f(x) = f(Rx)$.
We will use the following mixing lemma whose proof is an easy exercise

Lemma 1. *Let* $(g, f) \in (C^1(T^d))^2$ *and* $R \in \mathbb{N}^*$ *then*

$$\left| \int_{T_1^d} g(x) S_R f(x) dx - \int_{T_1^d} g(x) dx \int_{T_1^d} f(x) dx \right| \leq (\|g'\|_\infty / R) \int_{T_1^d} |f| dx. \quad (50)$$

By Lemma 1 and the Cauchy–Schwartz inequality,

$$|E| \leq \frac{\|\nabla h_p\|_\infty R_{p-1}}{R_p} \int_0^1 |S_{R_p} H^{p-1}(x)| dx \leq \frac{K_1}{r_p} \sqrt{\mathrm{Var}(H^{p-1})}. \quad (51)$$

Combining this with (49) one obtains (46).

4.2. Anomalous mean squared displacement: Theorem 1.

4.2.1. Anomalous behavior from perpetual homogenization. Let y_t be the solution of (3). Define

$$n_{ef}(t) = \inf\{n \in \mathbb{N} : t \leq R_{n+1}^2 (\gamma_{n-1}/\gamma_{n+1})^2 2^{-3} K_1^{-2} (1 - \gamma_{\max}/\rho_{\min})^2\}. \quad (52)$$

$n_{ef}(t)$ will be the number of effective scales that one can consider *homogenized* in the estimation of the mean squared displacement at the time t. Indeed, we will show in Subsect. 4.2.2 that for $\rho_{\min} > C_{1,K_1,\gamma_{\min},\gamma_{\max}}$ and $t > C_{2,K_0,\gamma_{\min},\gamma_1,R_1}$ one has

$$(1/4) t \gamma_{n_{ef}(t)-1}^2 \leq \mathbb{E}[(y_t.e_2)^2] \leq C_{3,K_1,\gamma_{\min}} \gamma_{n_{ef}(t)+1}^2 t. \quad (53)$$

Combining this with the bounds (9) and (10) on R_n and γ_n one easily obtains Theorem 1.

4.2.2. Distinction between effective *and* drift *scales. Proof of Eq. (53).* By the Ito formula one has

$$y_t.e_2 = \omega_t.e_2 + \int_0^t \partial_1 h(\omega_s.e_1) \, ds. \quad (54)$$

And by the independence of $\omega_t.e_2$ from $\omega_t.e_1$ one obtains

$$\mathbb{E}[(y_t.e_2)^2] = t + \mathbb{E}\left[\left(\int_0^t \partial_1 h(\omega_s.e_1) ds\right)^2\right]. \quad (55)$$

Thus for all $p \in \mathbb{N}^*$, using $h = H^p + H^{p+1,\infty}$ one easily obtains (writing $H^p = H^{0,p}$)

$$\mathbb{E}[(y_t.e_2)^2] \geq t + \frac{1}{2} \mathbb{E}\left[\left(\int_0^t \partial_1 H^p(\omega_s.e_1) ds\right)^2\right] - \mathbb{E}\left[\left(\int_0^t \partial_1 H^{p+1,\infty}(\omega_s.e_1) ds\right)^2\right]$$

$$\leq t + 2\mathbb{E}\left[\left(\int_0^t \partial_1 H^p(\omega_s.e_1) ds\right)^2\right] + 2\mathbb{E}\left[\left(\int_0^t \partial_1 H^{p+1,\infty}(\omega_s.e_1) ds\right)^2\right].$$

$$(56)$$

Now we will bound the larger scales $\partial_1 H^{p+1,\infty}$ as drift scales, i.e. bound them by a constant drift using $\|\partial_1 H^{p+1,\infty}\|_\infty \le K_1(\gamma_{p+1}/R_{p+1})(1 - \gamma_{max}/\rho_{min})^{-1}$,

$$
\begin{aligned}
\mathbb{E}[(y_t.e_2)^2] &\ge t + \frac{1}{2}\mathbb{E}\Big[\Big(\int_0^t \partial_1 H^p(\omega_s.e_1)ds\Big)^2\Big] - \Big(\frac{tK_1\gamma_{p+1}}{R_{p+1}(1 - \gamma_{max}/\rho_{min})}\Big)^2 \\
&\le t + 2\mathbb{E}\Big[\Big(\int_0^t \partial_1 H^p(\omega_s.e_1)ds\Big)^2\Big] + 2\Big(\frac{tK_1\gamma_{p+1}}{R_{p+1}(1 - \gamma_{max}/\rho_{min})}\Big)^2.
\end{aligned}
\tag{57}
$$

Write $I_p = \mathbb{E}\Big[\big(\int_0^t \partial_1 H^p(\omega_s.e_1)ds\big)^2\Big]$. Since H^p is periodic, for t large enough, I_p should behave like $t\gamma_p^2$. Nevertheless since the ratios between the scales are bounded, to control the asymptotic lower bound of the mean squared displacement in (57) we will need a quantitative control of I_p that is sharp enough to show that the influence of the effective scales is not destroyed by the larger ones. This control is based on stochastic mixing inequalities and it will be shown in Subsect. 4.2.3 that for $\rho_{min} > 8K_1/(\gamma_{min} - 1)$ one has

$$
\begin{aligned}
I_p \le\, &t23\big(\gamma_{p-1}^2(1 - 1/\gamma_{min})^{-2} + K_0K_1\gamma_{p-1}(\gamma_p/r_p)(1 - 1/\gamma_{min})^{-1}\big) \\
&+ t^2 K_1^2(\gamma_p^2/R_p^2) + \sqrt{t}68\gamma_{p-1}\gamma_p R_{p-1}K_0^2(1 - 1/\gamma_{min})^{-1} \\
&+ 60K_0^2(1 - 1/\gamma_{min})^{-1}\gamma_{p-1}\gamma_p(R_p^2/r_p) + 16K_0^2\gamma_{p-1}^2 R_{p-1}^2
\end{aligned}
\tag{58}
$$

and

$$
\begin{aligned}
I_p \ge\, &t\gamma_{p-1}(\gamma_{p-1} - 16K_1\gamma_p/r_p) - \sqrt{t}68\gamma_{p-1}\gamma_p R_{p-1}K_0^2(1 - 1/\gamma_{min})^{-1} \\
&- 60K_0^2(1 - 1/\gamma_{min})^{-1}\gamma_{p-1}\gamma_p\frac{R_p^2}{r_p} - 8K_0^2\gamma_{p-1}^2 R_{p-1}^2(1 - 1/\gamma_{min})^{-2}.
\end{aligned}
\tag{59}
$$

Choosing $p = n_{ef}(t)$ given by (52) one obtains (53) from (58), (59) and (57) by straightforward computation under the assumption $\rho_{min} > C_{K_0, K_1, \gamma_{max}, \gamma_{min}}$. $\quad\square$

4.2.3. Influence of the intermediate *scale on the* effective *scales: Proof of the inequalities (59) and (58).* In this subsection we will prove the inequalities (59) and (58) by distinguishing the scale p as a perturbation scale, i.e. controlling its influence on the homogenization process over the scales $0, \dots, p - 1$. More precisely, writing b_t the Brownian motion $\omega_t.e_1$ one has

$$
I_p = I_{p-1} + \mathbb{E}\Big[\Big(\int_0^t \partial_1 H^{p,p}(b_s)ds\Big)^2\Big] + 2J_p
\tag{60}
$$

with

$$
J_p = \mathbb{E}\Big[\Big(\int_0^t \partial_1 H^{p,p}(b_s)ds\Big)\Big(\int_0^t \partial_1 H^{p-1}(b_s)ds\Big)\Big].
\tag{61}
$$

We will then control (60) by bounding $\partial_1 H^{p,p}$ by a constant drift to obtain

$$
\mathbb{E}\Big[\Big(\int_0^t \partial_1 H^{p,p}(b_s)ds\Big)^2\Big] \le t^2 K_1^2\gamma_p^2/R_p^2.
\tag{62}
$$

In Subsect. 4.2.4 we will control the homogenization process over the scales $0, \ldots, p-1$ and use our estimates on $D(\Gamma^{0,p-1})$ to obtain that for $\rho_{\min} > 8K_1/(\gamma_{\min} - 1)$,

$$I_{p-1} \leq t\gamma_{p-1}^2 20(1 - 1/\gamma_{\min})^2 + R_{p-1}^2\gamma_{p-1}^2 16K_0^2(1 - 1/\gamma_{\min})^2, \tag{63}$$

$$I_{p-1} \geq t\gamma_{p-1}^2 2(1 - 1/\gamma_{\min})^2 - R_{p-1}^2\gamma_{p-1}^2 8K_0^2(1 - 1/\gamma_{\min})^2. \tag{64}$$

In Subsect. 4.2.5 we will bound J_p is an error term by mixing stochastic inequalities to obtain

$$|J_p| \leq \gamma_{p-1}\gamma_p(1 - 1/\gamma_{\min})^{-1}\left(\sqrt{t}R_{p-1}34K_0^2 + (t/r_p)8K_1 + R_p^2 r_p^{-1}30K_0^2\right). \tag{65}$$

Combining (60), (62), (63), (64) and (65) one obtains (58) and (59). □

4.2.4. Control of the homogenization process over the *effective scales: Proof of Eqs. (63) and (64).* Writing for $m \leq p$,

$$\kappa^{m,p} = 1/R_p \int_0^{R_p} H^{m,p}(y)dy \tag{66}$$

and $\kappa^p = \kappa^{0,p}$ one obtains by the Ito formula

$$\int_0^t \partial_1 H^{p-1}(b_s)ds = 2\int_0^{b_t} (H^{p-1}(y) - \kappa^{p-1})dy - 2\int_0^t (H^{p-1}(b_s) - \kappa^{p-1})db_s. \tag{67}$$

Using the periodicity of H^p:

$$\left|\int_0^x \left(H^{p-1}(y) - \kappa^{p-1}\right)dy\right| \leq R_{p-1}K_0\gamma_{p-1}(1 - 1/\gamma_{\min})^{-1}. \tag{68}$$

Now we will show that for $\rho_{\min} > 8K_1/(\gamma_{\min} - 1)$ and $p \in \mathbb{N}^*$,

$$\mathbb{E}[\int_0^t (H^p(b_s) - \kappa^p)^2 \, ds] \leq (1 - 1/\gamma_{\min})^{-2}\left(t\gamma_p^2 5 + R_p^2\gamma_p^2 4K_0^2\right)$$
$$\geq t\gamma_p^2 2 - R_p^2\gamma_p^2 4K_0^2(1 - 1/\gamma_{\min})^{-2}. \tag{69}$$

Combining (67), (68) and (69) one obtains (63) and (64) by straightforward computation. The proof of (69) is based on standard homogenization theory: writing

$$f(x) = \int_0^x \left(H^p(y) - \kappa^p\right)^2 - \text{Var}(H^p)x, \tag{70}$$

$$\text{and} \quad g(x) = 2\left(\int_0^x f(y) \, dy - (x/R_p)\int_0^{R_p} f(y) \, dy\right). \tag{71}$$

One obtains by the Ito formula

$$\mathbb{E}[\int_0^t (H^p(b_s) - \kappa^p)^2 \, ds] = \text{Var}(H^p)t + \mathbb{E}[g(b_t)]. \tag{72}$$

Using the periodicity of g, one has $\|g\|_\infty \leq 4K_0^2\gamma_p^2 R_p^2(1 - 1/\gamma_{\min})^{-2}$. Combining this with the estimate (45) on $\text{Var}(H^p)$ one obtains (69) from (72). □

4.2.5. Control of the influence of the perturbation *scale: Proof of Eq. (65).* From Eqs. (61), (67),

$$\left| \int_0^x (H^{p-1}(y) - \kappa^{p-1}) dy \right| \le R_{p-1} \gamma_{p-1} K_0 (1 - 1/\gamma_{\min})^{-1} \tag{73}$$

and

$$\int_0^t \partial_1 H^{p,p}(\omega_s . e_1) ds = 2 \int_0^{b_t} (H^{p,p}(y) - \kappa^{p,p}) dy - 2 \int_0^t (H^{p,p}(b_s) - \kappa^{p,p}) \, db_s. \tag{74}$$

One obtains by using straightforward computation (using the Cauchy–Schwartz inequality) that

$$|J_p| \le 2\gamma_{p-1}\gamma_p R_{p-1} K_0^2 (1 - 1/\gamma_{\min})^{-1} \sqrt{t} + 4|J_{p,2}| + 2|J_{p,3}| \tag{75}$$

with

$$J_{p,2} = \mathbb{E}\left[\int_0^t (H^{p-1}(b_s) - \kappa^{p-1})(H^{p,p}(b_s) - \kappa^{p,p}) \, ds \right] \tag{76}$$

and

$$J_{p,3} = \mathbb{E}\left[\int_0^t \partial_1 H^{p-1}(b_s) ds \int_0^{b_t} (H^{p,p}(y) - \kappa^{p,p}) dy \right]. \tag{77}$$

We will show in Subsect. 4.2.6 that the ratio r_p allows a stochastic separation between the scales $0, \dots, p-1$ and p reflected by the following inequality:

$$|J_{p,2}| \le \gamma_{p-1}\gamma_p K_0 (1 - 1/\gamma_{\min})^{-1} (8K_0 R_{p-1}\sqrt{t} + 2t K_1/r_p). \tag{78}$$

Using Proposition 1 (that we will prove Subsect. 4.2.7) with $G(x) = \int_0^x (H^{p,p}(y) - \kappa^{p,p}) dy$, $r = r_p$ and $f(r_p x) = H^{p-1}(x) - k^{p-1}$ that

$$|J_{p,3}| \le \gamma_{p-1}\gamma_p R_p^2 r_p^{-1} 15 K_0^2 (1 - 1/\gamma_{\min})^{-1}. \tag{79}$$

Combining (75), (78) and (79) one obtains (65). □

4.2.6. Stochastic separation between scales: Proof of Eq. (78). Writing for $x \in \mathbb{R}$,

$$g(x) = \int_0^x (H^{p-1}(y) - \kappa^{p-1})(H^{p,p}(y) - \kappa^p) \, dy, \tag{80}$$

one obtains by Ito formula

$$2\mathbb{E}[\int_0^{b_s} g(y) dy] = J_{p,2}. \tag{81}$$

Using the functional mixing Lemma 1 one obtains easily that

$$|g(x)| \le 2K_0 \gamma_{p-1} (1 - 1/\gamma_{\min})^{-1} (4\gamma_p K_0 R_{p-1} + |x| K_1 (\gamma_p/r_p)). \tag{82}$$

Combining this with (81) one obtains (78). □

4.2.7. Stochastic mixing: Proof of Proposition 1. By the scaling law of the Brownian motion

$$\mathbb{E}\left[G(b_t/R)\int_0^t (1/R)\partial_1 f(rb_s/R)\,ds\right] = R\mathbb{E}\left[G(b_{\frac{t}{R^2}})\int_0^{\frac{t}{R^2}} \partial_1 f(rb_s)\,ds\right]$$

and it is sufficient to prove the proposition assuming $R = 1$. Let us write

$$I = \mathbb{E}\left[G(b_t)\int_0^t \partial_1 f(rb_s)\,ds\right]. \tag{83}$$

We will prove Proposition 1 by expanding (83) on the the Fourier decompositions of f and G (written f_k and G_k) and controlling trigonometric functions,

$$f(x) = \sum_{k\in\mathbb{Z}} f_k e^{ik2\pi x}. \tag{84}$$

Write for $k, m \in \mathbb{Z}$,

$$J_{k,m} = \int_0^t \mathbb{E}\left[e^{ikr2\pi b_s}e^{im2\pi b_t}\right]ds. \tag{85}$$

By straightforward computation

$$\begin{aligned}
J_{k,m} &= \int_0^t e^{-(2\pi)^2\frac{(kr+m)^2}{2}s-(2\pi)^2\frac{m^2}{2}(t-s)} \\
&= e^{-(2\pi)^2\frac{m^2}{2}t}\frac{1-e^{-(2\pi)^2(\frac{(kr)^2}{2}+krm)t}}{(2\pi)^2(\frac{(kr)^2}{2}+krm)}
\end{aligned} \tag{86}$$

(in last fraction of the above equation, if the denominator is equal to 0, we consider it as a limit to obtain the exact value t).

Now

$$I = \sum_{k,m\in\mathbb{Z}^2} J_{km}ik2\pi f_k G_m. \tag{87}$$

Thus since $f_0 = G_0 = 0$, by the Cauchy–Schwartz inequality

$$|I| \le \sum_{k\in\mathbb{Z}^*} |f_k|\left(\sum_{m\in\mathbb{Z}}(2\pi)^2 m^2|G_m|^2\right)^{\frac{1}{2}} J_k^{\frac{1}{2}}. \tag{88}$$

with

$$J_k = \sum_{m\in\mathbb{Z}^*} \frac{1}{m^2}\left(\frac{1-e^{-(2\pi)^2(\frac{(kr)^2}{2}+krm)t}}{(\frac{k^2r^2}{2}+krm)(2\pi)^2}\right)^2 e^{-(2\pi)^2 m^2 t}. \tag{89}$$

Using $(1-e^{-tx})/x \le 3t$ for $x > 0$ and the fact that the minimum of $k^2r/2 + km$ is reached for $m_0 \sim kr/2$, we obtain

$$\begin{aligned}
J_k &\le \sum_{m\in\mathbb{Z}^*} e^{-(2\pi)^2 m^2 t^2}\left(\frac{4t}{k^2r^2}+2\sum_{m\in\mathbb{Z}^*}\frac{1}{m^2}\left(\frac{1}{krm(2\pi)^2}\right)^2\right) \\
&\le 4/r^2.
\end{aligned} \tag{90}$$

Observing that $\|G'\|^2_{L^2(\mathbb{T}^1)} = (2\pi)^2 \sum_{m\in\mathbb{Z}} m^2 |G_m|^2$ one obtains from (88) and (90)

$$|I| \leq \|G'\|_{L^2(\mathbb{T}^1)} (2/r) \sum_{k\in\mathbb{Z}^*} |f_k|, \tag{91}$$

which, using the Cauchy–Schwartz inequality leads to

$$|I| \leq \|G'\|_{L^2(\mathbb{T}^1)} \|f\|_{L^2(\mathbb{T}^1)} 2/r. \tag{92}$$

□

5. Proofs Under Hypothesis 2

In this section we will prove Theorem 3 under Hypothesis 2. Observe that it is sufficient to prove Eq. (29) under Hypothesis 2, (28) and (31).

We will use the same notations as in Sect. 4. Let us observe that from Eq. 55 one obtains

$$\mathbb{E}[(y_t.e_2)^2] = t + \mathbb{E}[X^2(b,0,t)] + \mathbb{E}[Y^2(b,0,t)] + 2\mathbb{E}[X(b,0,t)Y(b,0,t)] \tag{93}$$

with for $p \in \mathbb{N}$ and $b_s = \omega_s.e_1$,

$$X(b,0,t) = \int_0^t \partial_1 H^{0,p}(b_s)\, ds \tag{94}$$

and

$$Y(b,0,t) = \int_0^t \partial_1 H^{p+1,\infty}(b_s)\, ds. \tag{95}$$

We will prove in Subsect. 5.1 the following lemma (which is the core of the proof of Theorem 3) which gives quantitative control on decorrelation between the drifts associated with the small scales and those associated to the larger ones.

Lemma 2. *For* $t > R_p^2$,

$$\begin{aligned} \left|\mathbb{E}[X(b,0,t)Y(b,0,t)]\right| &\leq \left(t^{3/2}R_{p+1}^2 R_{p+2}^{-2} + 8t R_p\right) R_{p+1}^{-2} R_p \gamma_p \gamma_{p+1} \\ &\quad 12K_0 K_2 \left(1-\gamma_{\min}^{-1}\right)^{-1}\left(1-\gamma_{\max}/\rho_{\min}^2\right)^{-1}. \end{aligned} \tag{96}$$

Now by standard homogenization as has been done in Subsect. 4.2.4 it is easy to prove that

$$\mathbb{E}[X(b,0,t)^2] \leq (R_p^2+t)\gamma_p^2 8K_0^2\left(1-\gamma_{\min}^{-1}\right)^{-1} \tag{97}$$

and

$$\mathbb{E}[X(b,0,t)^2] \geq 2t\, \mathrm{Var}(H^p) - 8R_p^2 \gamma_p^2 K_0^2\left(1-\gamma_{\min}^{-1}\right)^{-1}. \tag{98}$$

Now we will give in Lemma 3 (proven in Subsect. 5.2) the exponential rate of divergence of $\mathrm{Var}(H^p)$.

Lemma 3. *For* $\gamma_{\min} > 2$ *one has*

$$\gamma_p^2\left(1-(\gamma_{\min}-1)^{-1}\right)^2 \leq \mathrm{Var}(H^p) \leq \gamma_p^2\left(1-\gamma_{\min}^{-1}\right)^{-2}. \tag{99}$$

The proof of Lemma 3 is different from that of Theorem 2 and is based on the domination of the influence of the biggest scale over the smaller ones (which is ensured by $\gamma_{\min} > 2$, which is also necessary in the case $\rho_{\min} > \gamma_{\max}$ to avoid cocycles).

Combining (99) with (98) one obtains that

$$\mathbb{E}[X(b, 0, t)^2] \geq 2t\gamma_p^2\big(1 - (\gamma_{\min} - 1)^{-1}\big)^2 - 8R_p^2\gamma_p^2 K_0^2\big(1 - \gamma_{\min}^{-1}\big)^{-1}. \qquad (100)$$

Now let us observe that from (93) and the Cauchy–Schwartz inequality one obtains

$$\begin{aligned}
\mathbb{E}\big[(y_t.e_2)^2\big] &\leq t + 2\mathbb{E}\big[X^2(b, 0, t)\big] + 2\mathbb{E}\big[Y^2(b, 0, t)\big] \\
&\geq t + \mathbb{E}\big[X^2(b, 0, t)\big] - 2\big|\mathbb{E}[X(b, 0, t)Y(b, 0, t)]\big|.
\end{aligned} \qquad (101)$$

Combining (100), (97), (96), (101) and bounding $\mathbb{E}\big[Y^2(b, 0, t)\big]$ by

$$\begin{aligned}
\mathbb{E}\big[Y^2(b, 0, t)\big] &\leq \|\partial_1^2 H^{p+1,\infty}\|_\infty^2 t^2 \mathbb{E}[b_t^2] \\
&\leq \gamma_{p+1}^2 t^3 R_{p+1}^{-4}\Big(K_2(1 - \gamma_{\min}^{-1})^{-1}(1 - \gamma_{\max}/\rho_{\min}^2)^{-1}\Big)^2,
\end{aligned} \qquad (102)$$

one obtains that for $t > R_p^2$, $\gamma_{\min} > 2$ and $\rho_{\min}^2 > \gamma_{\max}$ that

$$\mathbb{E}[(y_t.e_2)^2] \leq 32(R_p^2 + t)\gamma_p^2 K_0^2 + 8\gamma_{p+1}^2 t^3 R_{p+1}^{-4}\Big(K_2(1 - \gamma_{\max}/\rho_{\min}^2)^{-1}\Big)^2 \qquad (103)$$

and

$$\begin{aligned}
\mathbb{E}[(y_t.e_2)^2] \geq{}& 2t\gamma_p^2\big(1 - (\gamma_{\min} - 1)^{-1}\big)^2 - 16R_p^2\gamma_p^2 K_0^2 \\
&- \big(t^{3/2}R_{p+1}^2 R_{p+2}^{-2} + 8t R_p\big)R_{p+1}^{-2}R_p\gamma_p\gamma_{p+1}48K_0 K_2(1 - \gamma_{\max}/\rho_{\min}^2)^{-1}.
\end{aligned} \qquad (104)$$

It follows (104) and (103) that for

$$\begin{aligned}
16(1 + K_0^2)\big(1 - (\gamma_{\min} - 1)^{-1}\big)^{-2}R_p^2 &< t \\
&\leq 16(1 + K_0^2)\big(1 - (\gamma_{\min} - 1)^{-1}\big)^{-2}R_{p+1}^2.
\end{aligned} \qquad (105)$$

One has

$$\begin{aligned}
\mathbb{E}[(y_t.e_2)^2] \geq{}& t\gamma_p^2\Big(\big(1 - (\gamma_{\min} - 1)^{-1}\big)^2 - \gamma_{\max}\rho_{\min}^{-2}400K_0 K_2(1 - \gamma_{\max}/\rho_{\min}^2)^{-1} \\
&\times \big(1 + (1 + K_0)\rho_{\min}^{-1}(\gamma_{\min} - 1)(\gamma_{\min} - 2)^{-1}\big)\Big)
\end{aligned} \qquad (106)$$

and

$$\begin{aligned}
\mathbb{E}[(y_t.e_2)^2] \leq{}& t\gamma_p^2(1 + K_0^2) \\
&\times \Big(1 + 2050\gamma_{\max}^2\big(1 - (\gamma_{\min} - 1)^{-1}\big)^{-4}K_2^2(1 - \gamma_{\max}/\rho_{\min}^2)^{-2}\Big),
\end{aligned} \qquad (107)$$

which proves Eq. (29) under Hypothesis 2, (28) and (31) and thus Theorem 3.

5.1. Proof of Lemma 2. Let us introduce Z a random variable independent from b_s and taking its values in $[0, R_p]$ uniformly with respect to the Lebesgue measure. First, let us observe that by the periodicity of H^p,

$$\mathbb{E}[X(b+Z, 0, t)Y(b, 0, t)] = 0. \tag{108}$$

Next, define

$$\tau = \inf\{s > 0 : |b_s + Z| = 0 \text{ or } |b_s + Z| = R_p\}.$$

Now we will decompose $X(b + Z, 0, t)$ and $Y(b, 0, t)$ as follows:

$$X(b + Z, 0, t) = X(b + Z, 0, \tau) + X(b + Z, \tau, t), \tag{109}$$

$$Y(b, 0, t) = Y(b, 0, \tau) + Y(b, \tau, t). \tag{110}$$

Let b' be a BM independent from b and Z with the same law as b. Combining the decompositions (109) and (110) with the strong Markov property and the periodicity of H^p one obtains that

$$\begin{aligned}
\mathbb{E}[X(b + Z, 0, t)Y(b, 0, t)] &= \mathbb{E}[X(b + Z, 0, \tau \wedge t)Y(b, 0, \tau \wedge t)] \\
&\quad + \mathbb{E}[X(b', 0, (t - \tau)_+)Y(b' + b_\tau, 0, (t - \tau)_+)] \\
&\quad + \mathbb{E}[X(b + Z, 0, \tau \wedge t)Y(b, \tau \wedge t, t)] \\
&\quad + \mathbb{E}[X(b + Z, \tau \wedge t, t)Y(b, 0, \tau \wedge t)],
\end{aligned} \tag{111}$$

where we have used

$$\mathbb{E}[X(b', 0, (t - \tau)_+)Y(b' + b_\tau, 0, (t - \tau)_+)] = \mathbb{E}[X(b + Z, \tau \wedge t, t)Y(b, \tau \wedge t, t)].$$

Using a similar decomposition for $\mathbb{E}[X(b, 0, t)Y(b, 0, t)]$ and subtracting (111) combined with (108) one obtains that

$$\mathbb{E}[X(b, 0, t)Y(b, 0, t)] = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7. \tag{112}$$

with

$$I_1 = \mathbb{E}\big[X(b', 0, (t - \tau)_+)\big(Y(b', 0, (t - \tau)_+) - Y(b' + b_\tau, 0, (t - \tau)_+)\big)\big], \tag{113}$$

$$I_2 = \mathbb{E}\big[X(b', (t - \tau)_+, t)\big(Y(b', (t - \tau)_+, t)\big], \tag{114}$$

$$I_3 = -\mathbb{E}\big[X(b + Z, 0, \tau \wedge t)Y(b, 0, \tau \wedge t)\big], \tag{115}$$

$$I_4 = -\mathbb{E}\big[X(b + Z, 0, \tau \wedge t)Y(b, \tau \wedge t, t)\big], \tag{116}$$

$$I_5 = -\mathbb{E}\big[X(b + Z, \tau \wedge t, t)Y(b, 0, \tau \wedge t)\big], \tag{117}$$

$$I_6 = \mathbb{E}\big[X(b', 0, (t - \tau)_+)Y(b', (t - \tau)_+, t)\big], \tag{118}$$

$$I_7 = \mathbb{E}\big[Y(b', 0, (t - \tau)_+)X(b', (t - \tau)_+, t)\big]. \tag{119}$$

We shall use standard homogenization techniques to estimate the influence of the effective scales in those terms. Let us observe that by the Ito formula one has a.s. for all $t > 0$,

$$X(b, 0, t) = \chi^P(b_t) - 2 \int_0^t \left(H^P(b_s) - \kappa^P \right) db_s \tag{120}$$

with (using the notation introduced in (66))

$$\chi^P(x) = 2 \int_0^x H^P(y) dy - x 2 \kappa^P. \tag{121}$$

Next we will control the larger scales by bounding the growth of their drift. Indeed using the uniform control on the second derivate of the functions h_n in Hypothesis 2 one has

$$\|\partial_1^2 H^{p+1,\infty}\|_\infty \le K_2 \gamma_{p+1} R_{p+1}^{-2} (1 - \gamma_{max}/\rho_{min}^2)^{-1}. \tag{122}$$

Using the Cauchy–Schwartz inequality in (114), (120) and (122) one obtains that

$$|I_2| \le \left(\mathbb{E}\left[X(b', 0, (t - \tau)_+)^2 \right] \right)^{\frac{1}{2}} \left(\mathbb{E}\left[(Y(b', (t - \tau)_+, t))^2 \right] \right)^{\frac{1}{2}}$$

$$\le (1 - \gamma_{min}^{-1})^{-1} \left(4K_0^2 \gamma_p^2 R_p^2 + 4K_0^2 \gamma_p^2 t \right)^{\frac{1}{2}} \|\partial_1^2 H^{p+1,\infty}\|_\infty \left(\mathbb{E}\left[|b_t'|^2 \tau^2 \right] \right)^{\frac{1}{2}}.$$

Thus for

$$t > R_p^2, \tag{123}$$

$$|I_2| \le \sqrt{t} \gamma_p \gamma_{p+1} R_p^3 R_{p+1}^{-2} 12 K_0 K_2 (1 - \gamma_{min}^{-1})^{-1} (1 - \gamma_{max}/\rho_{min}^2)^{-1}. \tag{124}$$

Similarly using (120) one obtains that

$$|I_3| \le \gamma_p \gamma_{p+1} R_p^4 R_{p+1}^{-2} 12 K_0 K_2 (1 - \gamma_{min}^{-1})^{-1} (1 - \gamma_{max}/\rho_{min}^2)^{-1}. \tag{125}$$

To estimate I_4 we use the conditional independence

$$I_4 = - \mathbb{E}\left[1_{\tau \wedge t < t} \mathbb{E}\left[X(b + Z, 0, \tau \wedge t) | \tau \wedge t \right] \mathbb{E}\left[Y(b, \tau \wedge t, t) | \tau \wedge t \right] \right]$$
$$- \mathbb{E}\left[1_{\tau \wedge t = t} \mathbb{E}\left[X(b + Z, 0, \tau \wedge t) | \tau \wedge t \right] \mathbb{E}\left[Y(b, \tau \wedge t, t) | \tau \wedge t \right] \right]. \tag{126}$$

Since $Y(b, t, t) = 0$ the second term in (126) is null. Moreover conditionally on the event $\tau < t$, $b_{\tau \wedge t} + Z$ with equal chances $1/2$ is equal to 0 or R_p, so using (120) one obtains that on $\{\tau \wedge t < t\}$, $\mathbb{E}\left[X(b + Z, 0, \tau \wedge t) | \tau \wedge t \right] = 0$, which leads to

$$I_4 = 0. \tag{127}$$

Similarly, using conditional independence (120) and (122) one easily obtains

$$|I_5| \le \gamma_p \gamma_{p+1} R_p^4 R_{p+1}^{-2} 12 K_0 K_2 (1 - \gamma_{min}^{-1})^{-1} (1 - \gamma_{max}/\rho_{min}^2)^{-1}, \tag{128}$$

$$|I_6| \le \sqrt{t} \gamma_p \gamma_{p+1} R_p^3 R_{p+1}^{-2} 12 K_0 K_2 (1 - \gamma_{min}^{-1})^{-1} (1 - \gamma_{max}/\rho_{min}^2)^{-1}. \tag{129}$$

To estimate I_7 let us observe that from (120) and the spectral gap associated to the Laplacian on the torus (which is equivalent to the Poincaré inequality, see [23]) one has

$$|\mathbb{E}[X(b', (t - \tau)_+, t)|t - \tau]| \leq |\chi^P(b_t) - \chi^P(b_{(t-\tau)_+})| \tag{130}$$
$$\leq 12e^{-R_p^{-2}/2(t-\tau)_+} K_0(1 - \gamma_{\min}^{-1})^{-1}\gamma_p R_p.$$

Thus using conditional independence and distinguishing the events $\tau > t/2$ (whose probability is bounded using the spectral gap) and $\tau \leq t/2$, one obtains

$$|I_7| \leq t^{3/2}e^{-tR_p^{-2}/4}\gamma_p\gamma_{p+1}16R_pR_p^{-2}K_0K_2(1 - \gamma_{\min}^{-1})^{-1}(1 - \gamma_{\max}/\rho_{\min}^2)^{-1} \tag{131}$$
$$\leq \gamma_p\gamma_{p+1}16R_p^4R_p^{-2}K_0K_2(1 - \gamma_{\min}^{-1})^{-1}(1 - \gamma_{\max}/\rho_{\min}^2)^{-1}.$$

To estimate I_1 we will distinguish the scale $p + 1$ as an intermediate scale. First observe that

$$Y(b', 0, (t - \tau)_+) - Y(b' + b_\tau, 0, (t - \tau)_+) = 2\gamma_{p+1}J_1 + 2J_2 \tag{132}$$

with

$$J_2 = \int_0^{(t-\tau)_+} \left(\partial_1 H^{p+2,\infty}(b_s') - \partial_1 H^{p+2,\infty}(b_s' + b_\tau)\right)ds \tag{133}$$

and

$$J_1 = \int_0^{(t-\tau)_+} \left(\partial_1 h_{p+1}(b_s') - \partial_1 h_{p+1}(b_s' + b_\tau)\right)ds. \tag{134}$$

Using $|b_\tau| \leq R_p$ and (122), one obtains that for $t > R_p$,

$$\left|\mathbb{E}\left[X(b', 0, (t - \tau)_+)J_2\right]\right|$$
$$\leq t^{3/2}R_p\gamma_p\gamma_{p+1}R_{p+2}^{-2}2K_0K_2(1 - \gamma_{\min}^{-1})^{-1}(1 - \gamma_{\max}/\rho_{\min}^2)^{-1}. \tag{135}$$

Now using the Ito formula one has

$$J_1 = \int_0^{b_\tau} h_{p+1}(x)dx - \int_{b_{(t-\tau)_+}'}^{b_{(t-\tau)_+}' + b_\tau} h_{p+1}(x)dx$$
$$- \int_0^{(t-\tau)_+} \left(h_{p+1}(b_s') - h_{p+1}(b_s' + b_\tau)\right)db_s' \tag{136}$$

Thus observing that (τ, b_τ) has the same law as $(\tau, -b_\tau)$ one obtains that

$$\mathbb{E}\left[X(b', 0, (t - \tau)_+)J_1\right]$$
$$= \mathbb{E}\left[X(b', 0, (t - \tau)_+)\int_0^{b_\tau}\left(h_{p+1}(x) - h_{p+1}(-x)\right)/2\,dx\right]$$
$$+ \mathbb{E}\left[X(b', 0, (t - \tau)_+)\int_{b_{(t-\tau)_+}'}^{b_{(t-\tau)_+}' + b_\tau}\left(h_{p+1}(x) - h_{p+1}(x - b_\tau)\right)/2\,dx\right] \tag{137}$$
$$- \mathbb{E}\left[X(b', 0, (t - \tau)_+)\int_0^{(t-\tau)_+}\left(h_{p+1}(b_s') - h_{p+1}(b_s' + b_\tau)\right)db_s'\right].$$

It follows from (137) that for $t > R_p$,

$$
\begin{aligned}
&\left|\mathbb{E}\big[X(b', 0, (t-\tau)_+)\gamma_{p+1}J_1\big]\right| \\
&\qquad \leq 6K_0\gamma_p t R_p^2 \gamma_{p+1} R_{p+1}^{-2} K_2 \big(1-\gamma_{\min}^{-1}\big)^{-1}\big(1-\gamma_{\max}/\rho_{\min}^2\big)^{-1}.
\end{aligned}
\tag{138}
$$

Thus from (138) and (135) we deduce that

$$
|I_1| \leq \big(t^{3/2}R_{p+2}^{-2} + t R_p R_{p+1}^{-2}\big) R_p \gamma_p \gamma_{p+1} 12 K_0 K_2 \big(1-\gamma_{\min}^{-1}\big)^{-1}\big(1-\gamma_{\max}/\rho_{\min}^2\big)^{-1}.
\tag{139}
$$

Combining (139), (124), (125), (127), (128), (129), (131) and (96) we have obtained that for $t > R_p^2$, Eq. (96) is valid.

5.2. Proof of Lemma 3. Observe that (using the notation introduced in (66))

$$
\mathrm{Var}(H^p) = \mathrm{Var}(H^{p-1}) + \gamma_p^2 + 2R_p^{-1}\int_0^{R_p} \big(H^{p-1}(x) - \kappa^{0,p-1}\big)\big(h_p(x) - \kappa^{p,p}\big).
\tag{140}
$$

It follows by the Cauchy–Schwartz inequality and the normalization condition (7) that

$$
\begin{aligned}
\mathrm{Var}(H^{p-1}) + \gamma_p^2 + 2\gamma_p\,\mathrm{Var}(H^{p-1})^{\frac{1}{2}} &\geq \mathrm{Var}(H^p) \\
&\geq \mathrm{Var}(H^{p-1}) + \gamma_p^2 - 2\gamma_p\,\mathrm{Var}(H^{p-1})^{\frac{1}{2}},
\end{aligned}
$$

from which we deduce

$$
\mathrm{Var}(H^p)^{\frac{1}{2}} \leq \gamma_p + \mathrm{Var}(H^{p-1})^{\frac{1}{2}}
\tag{141}
$$

and

$$
\mathrm{Var}(H^p)^{\frac{1}{2}} \geq \gamma_p - \mathrm{Var}(H^{p-1})^{\frac{1}{2}}.
\tag{142}
$$

From (141) one obtains by induction that

$$
\mathrm{Var}(H^p)^{\frac{1}{2}} \leq \gamma_p(1-\gamma_{\min}^{-1})^{-1}.
\tag{143}
$$

Combining (142) with (143) one obtains that

$$
\mathrm{Var}(H^p)^{\frac{1}{2}} \geq \gamma_p\big(1-(\gamma_{\min}-1)^{-1}\big),
\tag{144}
$$

and observe that $1 - (\gamma_{\min}-1)^{-1} > 0$ for $\gamma_{\min} > 2$.

References

1. Avellaneda, M.: Homogenization and renormalization, the mathematics of multi-scale random media and turbulent diffusion. *Lectures in Applied Mathematics*, Volume **31**, 1996, pp. 251–268
2. Avellaneda, M., Majda, A.: Homogenization and renormalization of multiple-scattering expansions for green functions in turbulent transport. In: *Composite Media and Homogenization Theory*, Volume **5** of *Progress in Nonlinear Differential Equations and Their Applications*, 1987, pp. 13–35
3. Avellaneda, M., Majda, A.: Mathematical models with exact renormalization for turbulent transport. Commun. Math. Phys. **131**, 381–429 (1990)
4. Batchelor, G.: Diffusion in a field of homogeneous turbulence ii. The relative motion of particles. Proc. Cambridge Philos. Soc. **48**, 345 (1952)
5. Ben Arous, G., Owhadi, H.: Multi-scale homogenization with bounded ratios and anomalous slow diffusion. Preprint available at http://dmawww.epfl.ch/∼owhadi/ (2001)
6. Bhattacharya, R.: Multiscale diffusion processes with periodic coefficients and an application to solute transport in porous media. Ann. Appl. Probab. **9**(4), 951–1020 (1999)
7. Bhattacharya, R., Denker, M., Goswami, A.: Speed of convergence to equilibrium and to normality for diffusions with multiple periodic scales. Stoch. Processes and their Appl. **80**, 55–86 (1999)
8. Fannjiang, A.: Anomalous diffusion in random flows. In: *Mathematics of Multiscale Materials – The IMA volumes in mathematics and its applications*, Volume **99**, 1999, pp. 81–99
9. Fannjiang, A., Komorowski, T.: Fractional brownian motion limit for motions in turbulence. Ann. of Appl. Prob. **10**(4) (2001)
10. Fannjiang, A., Papanicolaou, G.: Convection enhanced diffusion for periodic flows. SIAM J. Appl. Math. **54**, 333–408 (1994)
11. Fannjiang, A., Papanicolaou, G.: Diffusion in turbulence. Probab. Theory Related Fields **105**(3), 279–334 (1996)
12. Furtado, F., Glimm, J., Lindquist, B., Pereira, F.: Multiple length scale calculus of mixing lenght growth in tracer floods. Kovarik, F., ed., *Proceeding of the emerging technologies conference, Houston TX*, Institute for Improved Oil Recovery, 1990, pp. 251–259.
13. Furtado, F., Glimm, J., Lindquist, B., Pereira, F., Zhang, Q.: Time dependent anomalous diffusion for flow in multi-fractal porous media. Verheggan, T., editor, *Proceeding of the workshop on numerical methods for simulation of multiphase and complex flow*. New York: Springer Verlag, 1991, pp. 251–259.
14. Gaudron, G.: Scaling laws and convergence for the advection-diffusion equation. Ann. of Appl. Prob. **8**, 649–663 (1998)
15. Glimm, J., Lindquist, B., Pereira, F., Peierls, R.: The multi-fractal hypothesis and anomalous diffusion. Mat. Apl. Comput. **11**(2), 189–207 (1992)
16. Glimm, J., Zhang, Q.: Inertial range scaling of laminar shear flow as a model of turbulent transport. Commun. Math. Phys. **146**, 217–229 (1992)
17. Isichenko, M., Kalda, J.: Statistical topography. ii. Two-dimensional transport of a passive scalar. J. Nonlinear Sci. **1**, 375–396 (1991)
18. Kolmogorov, A.: The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers. Dokl.Akad.Nauk SSSR **4**(30), 9–13 (1941)
19. Komorowski, T., Olla, S.: On homogenization of time dependent random flows. To appear in Probability Theory and Related Fields (2000)
20. Landau, L., Lifshitz, E. *Fluid Mechanics, 2nd ed.* Moscow: MIR, 1984
21. Norris, J.: Long-time behaviour of heat flow: Global estimates and exact asymptotics. Arch. Rational Mech. Anal. **140**, 161–195 (1997)
22. Obukhov, A.: On the distribution of energy in the spectrum of turbulent flow. Izv. Akad. Nauk SSSR Ser. Georg. Geofiz. **5**, 453 (1941)
23. Olla, S.: *Homogenization of Diffusion Processes in Random Fields*. Ecole Polytechnique (1994). Cours Ecole Polytechnique
24. Owhadi, H.: *Anomalous diffusion and homogenization on an infinite number of scales*. Ph.D. thesis, EPFL – Swiss Federal Institute of Technology (2001). Thesis no 2340, January 2001. Available at http://dmawww.epfl.ch/∼owhadi/
25. Owhadi, H.: Anomalous slow diffusion from perpetual homogenization. Submitted (2001). Preprint available at http://dmawww.epfl.ch/∼owhadi/

26. Owhadi, H.: Richardson law linked with Kolmogorov law in turbulent diffusion. In preparation (2001)
27. Richardson, L. F.: Atmosphere diffusion shown on a distance-neighbour graph. Proc. R. Soc. London, Ser A **110**, 709 (1926)
28. Zhang, Q.: A multi-scale theory of the anomalous mixing length growth for tracer flow in heterogeneous porous media. J. Stat. Phys. **505**, 485–501 (1992)
29. Komorovski, T., Olla, S.: On the superdiffusive behavior of passive tracer with a Gaussian drift. Preprint available at http://www.cmap.polytechnique.fr/olla/pubolla.html (December 2001)

Communicated by P. Constantin