

# Towards a Packet-based Control Theory - Part II: Rate Issues

Ling Shi and Richard M. Murray

**Abstract**—Following the first part [16] of stabilizing a Linear Time Invariant (LTI) system in a packet-based network, this paper further studies various rate issues associated with networked control systems. Specifically, networks with finite bandwidth and packet drops and systems with finite control inputs are studied in details. Similar to what we did in [16], we assume that the LTI system is unstable but both controllable and observable. The state information is transmitted to the controller over a packet-based network. We also assume that there is a perfect link from the controller to the plant. However, we change the notion of the system being asymptotically stable to almost sure stable which is in a probabilistic framework. This is because packet drops by the network introduce unavoidable randomness. With the notion of almost sure stability, various rate results under different settings are given. Examples and simulations are provided to demonstrate the results.

## I. INTRODUCTION

Networked control systems have attracted considerable amount of interest to both control community and network and communication community in the past decade. As shown in the introduction of our first paper [16], the rich results from both the classical control theory and classical information theory are not enough to deal with the emerging applications exploring networked control structure. For details of classical information theory, readers are referred to [14], [4], [6] and for details of classical control theory, readers are referred to [10], [17], [3].

Networked control systems have lots of advantages when compared with classical feedback control systems. For example, they can reduce the system wiring, make the system easy to operate and maintain and later diagnose in case of malfunctioning, and increase system agility [21]. In spite of the great advantages that the networked control architecture brings, inserting a network in between the plant and the controller introduces many problems as well. Randomness that is inherent with the network breaks the many assumptions made by the control community. For instance, zero-delayed sensing and actuation, perfect information and synchronization are no longer guaranteed in the new system architecture. Those must be revisited and analyzed before any practical networked control systems are built. Several important issues that are pertained to networked control systems have been identified in [21]. Those mainly include that the transmission time is time varying and has different statistics depending on the underlying network models; network induced delays are unavoidable because of the

scheduling schemes; packet drops sometime occur because of network congestions; unlimited data rate is not possible because of finite bandwidth available.

In the past decade, many researchers have spent effort to those issues and a number of significant results were obtained and many are in progress. Many of the aforementioned issues are studied separately assuming other issues do not exist. Estimating the state and stabilization of the closed loop system over a digital communication channel which has a finite bandwidth was firstly introduced by Wong and Brockett [19], [20] and further pursued by [12], [7], [18], [8]. Following their spirit, Mitter in [11] described the need for a unified approach to control, communication, and computation. His former PhD students, Tatikonda [18] and Sahai [13] have presented some interesting results in the area of control under communication constraints. Specifically, Tatikonda in his PhD thesis gave a necessary and sufficient condition on the channel data rate such that a noiseless LTI system in the closed loop is asymptotically stable. He also gave rate results for stabilizing a noisy LTI system over the digital channel. Sahai in his thesis proposed the notion of anytime capacity to deal with real time estimation and control for a networked control system. The issue of determining the minimum bit-rate to achieve stability has also been studied by Nair [7] where he also considered a discrete communication channel in the feedback control loop.

As an extension of these works, we take initial steps towards developing a packet-based control theory. In our first paper [16], we concentrated on the stability problem of an unstable but both controllable and observable LTI system over a packet-based network. We assumed that between the observer and the controller, there was a encoder-decoder pair and a packet-based network which had a finite fixed data rate  $R$  bits/s (we change the rate symbol in this paper from [16] which should be clear from context). We also assumed that there was a perfect link from the controller to the plant (see Figure 1 for a system block diagram). We then gave a set of sufficient conditions under which the system can be stabilized for a given data rate  $R$ . In particular, these conditions yielded an upper bound on the minimum  $R$  for which the system can be stabilized. A recursive encoding-decoding scheme and an associated control law were proposed to achieve asymptotical stability for data rate exceeding this bound. An optimal bit allocation problem was investigated in which we asked about how to allocate the bits in a single packet for a subsystem such that the minimum upper bound on the data rate can be achieved. We finally formulated the optimal bit allocation problem as a Linear Matrix Inequality (LMI) optimization problem

Control and Dynamical Systems, California Institute of Technology; Pasadena, CA 91106.

Email: {shiling, murray}@cds.caltech.edu; Tel: (626) 395-2313, Fax: (626) 796-8914.

Work supported in part by NSF ITR grant CCR-0326554.

which can be solved efficiently using standard Semi-definite Programming (SDP) solvers.

In this paper, we consider the same rate issue but under different settings. Firstly, in our first paper, we assume there is no packet drops in the network. Here we do not assume this and we give corresponding rate result with a packet-dropping network. Secondly, systems with finite control inputs are considered. Then we combine the analysis together to link the rate issues, packet drops and finite control inputs into a single framework. We also changed the definition of stability from asymptotically stable in the sense of Lyapunov to almost sure stable. The definition of stability in the sense of Lyapunov can be found in [17] which is generally used for deterministic dynamical systems. The definition of stability in a probabilistic setting is not new. It is usually considered when there is inherent randomness in the system, for example, in the jump linear systems [5] or in stochastic hybrid systems [1]. In [5], the authors have given the most frequently seen definitions of stochastic stability. We use almost sure stability in our problem formulation which is rigorously defined in section III.

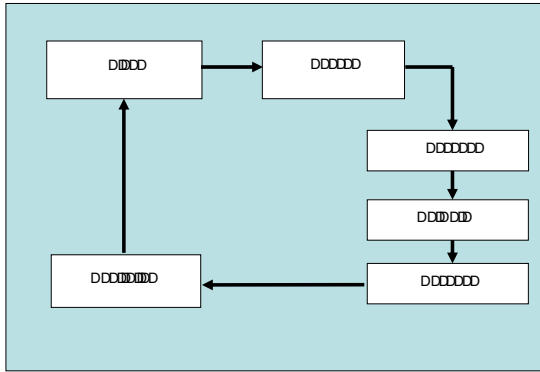


Fig. 1. Networked Control System Diagram

The paper is organized as follows. In section 2, we present the mathematical model of the closed loop system and state our assumptions. In section 3, we state the rate results under various circumstances. In particular, we consider the rate issues when packet drops from the network occur and only finite control inputs are available to the plant. Simulations are given in the end of this section. Following the main results on rate issues, conclusions and future work are given in the last section.

## II. SYSTEM MODELING AND PROBLEM SETTING

The plant in Figure 1 that we are interested in has the following dynamics

$$x_{k+1} = Ax_k + Bu_k, \quad (1)$$

$$y_k = Cx_k. \quad (2)$$

In the above equations,  $x_k \in \mathbb{R}^n$  is the state of the system,  $u_k \in \mathbb{R}^m$  is the control input and  $y_k \in \mathbb{R}^p$  is the output of the system. We also assume the initial condition  $x_k \in \mathbb{R}^n$  is bounded. The matrix  $A$  is assumed to be unstable to make the problem interesting, *i.e.*,  $A$  has at least one eigenvalue  $\lambda$  such that  $\text{Re}(\lambda) > 0$ . We ignore the cases where  $\lambda = 0$  can also cause the system to be unstable. Furthermore, we assume that the pair  $(A, B)$  is controllable and  $(C, A)$  is observable to make the problem tractable (see [3] for definitions of controllability and observability of LTI systems). The network in Figure 1 has finite data rate  $R$ , *i.e.*, the network can deliver  $R$  bits of information per discrete time step.

## III. RATE RESULTS FOR CLOSED LOOP STABILITY

In this section, we first give the definition of almost sure stability. After a brief review on the known results when there are no packet drops and arbitrary control input is allowed, we state the main rate results with packet drops and finite controls.

*Definition 1:* System (1) is called almost sure stable if

$$P\left\{\lim_{k \rightarrow \infty} \|x_k(x_0, \omega)\| = 0\right\} = 1,$$

where  $\omega$  is the underlying randomness for the closed loop system.

Asymptotical stability in the sense of Lyapunov requires that for any  $\varepsilon > 0$ , there exists a time  $T$ , such that for all  $k \geq T$ ,  $|x_k| \leq \varepsilon$ . For almost sure stability, however, it is allowed that  $x_k > \varepsilon$  for any  $k > 0$  and for any  $\varepsilon > 0$ .

### A. Arbitrary Control without Packet Drops

In Tatikonda's thesis work [18], various rate results were given for noiseless or noisy LTI systems over digital communication channels. We briefly state his main result that our results are based on in the theorem below.

*Theorem 2:* (Tatikonda [18]) Consider the discrete time system (1) in Figure 1 where the network is a digital communication channel with data rate  $R$ . Then a sufficient and necessary condition for the overall closed loop system to be asymptotically stable is that the minimum data rate of the digital channel  $R$  satisfies

$$R > R_{\min} = \sum_i \log \lambda_i(A),$$

where  $\lambda_i(A)$  are the unstable eigenvalues of  $A$  and  $\log$  has base 2.

**Proof:** See [18] for details.

QED

In his theorem, as Tatikonda was considering a digital channel in between the plant and the controller, there was no room for him to consider the delay or packet drop issues that are induced by a packet-based network. Our first paper [16] extended his result to a packet-based network which had various delay sources. Those include the transmission

delay because of the finite data rate, other maximum delay  $D$ , which could be the propagational delay, the queuing delay and so on. One of the theorems in [16] was provided below which used the equal bit allocation scheme.

*Lemma 3:* (Shi and Murray [16]) Consider the continuous version of system (1) in Figure 1 where the network is packet-based which has data rate  $R$  bits per second. Then a sufficient condition for the overall closed loop system to be asymptotically stable is that the minimum data rate  $R$  of the network satisfies

$$R > R_{\min} = \frac{l \log(|e^A|)}{\frac{l}{n} - 1 - D \log(|e^A|)},$$

where  $l$  is a single packet length,  $D$  is the other delays introduced by the network,  $|e^A|$  is the induced matrix norm of the matrix exponential  $e^A$  and  $\log$  has base 2.

**Proof:** See [16] for details.

QED

In [16], we did not touch the issue of packet drops which are inherent with a packet-based network due to the network congestions. Furthermore, similar to what Tatikonda did, we assumed that arbitrary control inputs were available. The last assumption is in general not true as typical physical systems have limited power constraints and hence only finite control inputs are allowed. In the rest of the paper, we give new rate results which properly take the new issues into account.

### B. Arbitrary Control with Packet Drops

In this section, we consider the problem of packet drops that are introduced by the network. The corresponding rate condition to guarantee almost sure stability is summarized in the following lemma. We consider the almost sure stability here because packet drops occur randomly which causes the classical notion of asymptotical stability not adequate.

*Lemma 4:* Assume packet arrivals are independently and identically distributed (i.i.d) with arrival rate  $\gamma$ , *i.e.*, a packet is received with probability  $\gamma$ . Consider the system (1) in the networked control setting (Figure 1). Then a necessary and sufficient condition on the network data rate  $R$  to guarantee almost sure stability for the closed loop system is that

$$R > R_{\min} = \frac{\sum_i \log \lambda_i(A)}{\gamma},$$

where  $\lambda_i(A)$ 's are the unstable eigenvalues of  $A$  and  $\log$  has base 2..

**Proof:** We give two proofs for the scalar case. One is from an information-theoretic point of view and the other one is a constructive approach where specific encoder and decoder are given. It is easy to prove the general case using an information-theoretic argument similar to the scalar case.

The first proof runs as follows. Let  $N$  to be the total number of time steps that the system has run, within which,

let  $n$  be the number of times that packets are received. Then for any given  $\epsilon > 0$ , from the weak law of large numbers,

$$P\left\{\lim_{N \rightarrow \infty} \left| \frac{n}{N} - \gamma \right| < \epsilon\right\} = 1,$$

or in other words,  $n = N\gamma$  is true with arbitrarily high probability. As a consequence,  $N\gamma R$  bits of information are received with arbitrarily high probability. During the  $N$  steps, the information loss is  $N \log a$  bits due to the initial uncertainty expanded by the system dynamics. On the other hand, the information gain is  $N\gamma R$  bits. Therefore the critical value of  $R$  follows the equation below

$$N\gamma R = N \log a,$$

which gives  $R_{\min} = \frac{\log a}{\gamma}$ . This makes sure that the system uncertainty is not growing almost surely. Any extra amount of additional information, *i.e.*, as long as  $R - R_{\min} > 0$ , it can be used to reduce the system uncertainty and hence make the state to converge to the origin. This completes the sufficiency part. The necessary part easily follows from Theorem 2, as  $R_{\min} = \log a$  is necessary to make system (1) asymptotically stable. To prove the general case, by Theorem 2, the information loss is now  $N \sum_i \log \lambda_i(A)$  instead of  $N \log a$ , hence the critical value of  $R$  follows from

$$N\gamma R = N \log \lambda_i(A),$$

which gives

$$R_{\min} = \frac{\sum_i \log \lambda_i(A)}{\gamma}.$$

Now we give a specific encoder and decoder which is similar to our preceding paper [16]. Let  $x_0 \leq 2^M$ . At time 1, the encoder encodes the most  $R$  significant bits of  $x_0$  and denote this  $R$  bits of information by  $\bar{x}_0$ . Hence we have

$$\varepsilon_0 = x_0 - \bar{x}_0 \leq 2^{M-R}.$$

If the decoder receives the packet, it can properly set the control to kill the known part  $\bar{x}_0$ , and hence make

$$x_1 \leq a\varepsilon_0 = a2^{M-R}.$$

Otherwise, it does nothing and leaves

$$x_1 \leq ax_0 = a2^M.$$

Following this control scheme, after  $N$  time steps, if  $n$  packets are received,

$$x_N \leq a^N 2^{M-nR}.$$

In order that  $x_N$  is not growing, it is required at least that

$$a^N = 2^{-nR},$$

or equivalently we write it as

$$R = \frac{N \log a}{n}.$$

From law of large numbers,  $\frac{n}{N} = \gamma$  is true with arbitrary high probability when  $N$  is sufficiently large. In other

words,  $R = \frac{\log a}{\gamma}$  guarantees that the state will not grow almost surely. Then similar to the first proof, any additional amount of information will bring down the state to the origin. Therefore,  $R_{\min} = \frac{\log a}{\gamma}$ .

QED

*Corollary 5:* From Lemma 4, the arrival rate  $\gamma$  such that the closed loop system in Figure 1 needs to have in order to achieve almost sure stability satisfies

$$\gamma > \gamma_{\min} = \frac{\sum_i \log \lambda_i(A)}{R},$$

provided that the network data rate is  $R$ . Or equivalently, the maximum packet drop rate  $\tau$  such that the closed loop system can tolerate satisfies

$$\tau < \tau_{\max} = 1 - \frac{\sum_i \log \lambda_i(A)}{R}.$$

*Remark 6:* If  $\gamma = 1$ ,  $R_{\min}$  is the same as in the Theorem 2 and the corresponding notion of almost sure stability is changed to asymptotical stability as the randomness is removed without packet drops. If  $\gamma$  tends to 0,  $R_{\min}$  tends to  $\infty$  which is as expected.

*Remark 7:* We assume here the packet arrivals are i.i.d which is in fact not strictly necessary. As long as we have a stable distribution of the arrival rate, the result still holds. This is similar to what we did in paper [15], where the packet arrivals form a markov process which has steady state distribution.

### C. Bounded Control without Packet Drops

To obtain all the above results, we assume the pair  $(A, B)$  is controllable. We implicitly assume the fact that arbitrary control inputs are allowed, *i.e.*,  $u_k$  could be unbounded. However, in real world applications, due to the limited power constraints, arbitrary control inputs are not possible. There are always constraints on the maximum size of the control inputs. Inspired by this fact, we study the rate issues with finite controls. The first lemma deals with system (1) alone with finite controls. The second lemma deals with finite control together with finite rate for the network. In both lemmas, packet drops are not considered. All the three factors are considered together in the next subsection.

*Lemma 8:* Consider the system (1) alone. Assume  $|u_k| \leq \bar{U}$  for all  $k$  and  $|x_0| \leq M$ . Then the system is asymptotically stabilizable if the following holds

$$M < M_{\max} = \frac{|B|}{|A| - 1} \bar{U},$$

where  $|A|$  and  $|B|$  are the induced matrix norm on  $A$  and  $B$ . Furthermore, this is also a necessary condition for scalar systems.

**Proof:** Let's consider a scalar system

$$x_{k+1} = ax_k + bu_k,$$

where  $|u_k| \leq \bar{U}$  and  $a > 1$ . Without loss of generality, assume  $x_0 = M$  and  $b > 0$ . Clearly if  $aM \leq b\bar{U}$ , or

$$M \leq \frac{b\bar{U}}{a},$$

the system can always be stabilized as we can just set  $u_0 = -\frac{a}{b}x_0$ . Otherwise, if  $aM > b\bar{U}$ , we set  $u_0 = -\bar{U}$ , hence  $x_1 = aM - b\bar{U}$ . Then if  $a(aM - b\bar{U}) \leq b\bar{U}$ , or

$$M \leq \frac{b\bar{U}}{a} \left(1 + \frac{1}{a}\right),$$

the system can also be stabilized as we can set  $u_1 = -\frac{a}{b}x_1$ .

Continuing this way, it is not hard to show that

$$M_{\max} = \frac{b\bar{U}}{a - 1}.$$

Now consider a general LTI system

$$x_{k+1} = Ax_k + Bu_k,$$

where the pair  $(A, B)$  is controllable and  $|u_k| \leq \bar{U}$ . We can proceed as follows.

$$\begin{aligned} |x_{k+1}| &= |Ax_k + Bu_k| \\ &\leq |Ax_k| + |Bu_k| \\ &\leq |A||x_k| + |B||u_k|, \end{aligned}$$

where  $|A|$  and  $|B|$  are the induced matrix norm of  $A$  and  $B$ . Now treat the above system as a scalar system, from the scalar analysis, a sufficient condition is then

$$M_{\max} = \frac{|B|}{|A| - 1} \bar{U}.$$

Note that  $A$  is unstable leads to the fact that  $|A| > 1$ .

QED

*Lemma 9:* Consider the system (1) in the networked setting as shown in Figure 1. Assume  $|u_k| \leq \bar{U}$  for all  $k$ ,  $|x_0| \leq M$  and the network has data rate  $R$ . Then as long as

$$M < M_{\max} = \frac{|B|}{|A| - 1} \bar{U}$$

and

$$R > R_{\min} = \sum_i \log \lambda_i(A),$$

the system is asymptotically stabilizable. In other words, the rate condition is independent of the bounds on the control input.

**Proof:** For simplicity, we give the proof here for the scalar system

$$x_{k+1} = ax_k + bu_k.$$

The proof can be easily extended to general LTI system. We use the same encoding and decoding scheme as we did in [16] as well as in the proof of Lemma 4. Assume  $x_0 > 0$  and  $b > 0$  without loss of generality. Every time step, the encoder encodes the most  $R$  significant bits of the state information. Let  $\bar{x}_k$  denotes this  $R$  bits version of the state

$x_k$ . Whenever  $a\bar{x}_k \leq b\bar{U}$ ,  $u_k$  can be set to be  $-\frac{a}{b}\bar{x}_k$ . In this case, the rate condition on  $R$ , *i.e.*,  $R > \log a$ , guarantees that  $x_{k+1} < x_k$ . Hence all the later known states can be killed which leads to the asymptotical convergence of the state. Otherwise, if  $a\bar{x}_k > b\bar{U}$ ,  $u_k = -\bar{U}$ . Compare this with the control scheme in Lemma 8, we see that every time step the maximum input needed is just  $\bar{U}$ . That is to say, we apply maximum all the time until at some time  $k < \infty$ , after which we start to apply less control and cause the state to converge asymptotically by the aforementioned analysis. Lemma 8 assures that such  $k < \infty$  exists.

QED

#### D. Bounded Control with Packet Drops

In this session, we put the three factors together, namely, the finite rate and packet drops from the network and finite control inputs from the plant. It turns out that we have a negative result, *i.e.*, the closed loop system is no longer almost sure stable if all the three factors exist. The following lemma precisely captures this.

**Lemma 10:** Consider the system (1) in the networked setting as shown in Figure 1. Assume  $|u_k| \leq \bar{U}$  for all  $k$  and  $|x_0| \leq M$ . If the packet arrival rate  $\gamma < 1$ , then no matter how large  $\bar{U}$  and  $R$  are, the closed loop system is not almost sure stable unless arbitrary control input is allowed.

**Proof:** Again, we prove here for the scalar system

$$x_{k+1} = ax_k + bu_k.$$

The proof can be easily extended to general cases. Assume  $x_0 > 0$  without loss of generality. For any  $\bar{U} < \infty$ , from Lemma 8,  $M$  must be bounded as well in order to achieve asymptotical stability, not mentioning almost sure stability. Hence there exists an  $N < \infty$  such that  $a^N x_0 > M$  as  $a > 1$ . As the probability of getting  $N$  consecutive packet drops is  $(1 - \gamma)^N > 0$ , *i.e.*, there is a positive probability such that the state can leave the stability region characterized by  $M$ , the system is then not almost sure stable.

QED

This completes all the main theorems and lemmas in this paper. In the following session, we provide some simulations to demonstrate the theoretical results.

#### E. Simulations

The following simulation results are all for the scalar system with dynamics

$$x_{k+1} = 2x_k + u_k.$$

The first two are examples for Lemma 4.

In Figure 2, the packet arrival rate  $\gamma = 0.7$ . According to Lemma 4,  $R_{\min} = \frac{\log a}{\gamma} = 1.4286$ . The actual network data rate  $R = 2$ , which satisfies  $R > R_{\min}$ , hence the closed loop system is almost sure stable which can be seen from the plot.

In Figure 3, the packet arrival rate  $\gamma = 0.5$ . According to Lemma 4,  $R_{\min} = \frac{\log a}{\gamma} = 2$ .  $R$  is still 2, however this time,

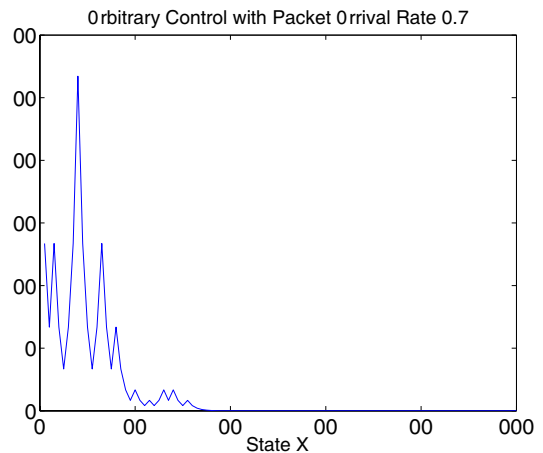


Fig. 2. Any Control with Rate Condition Satisfied

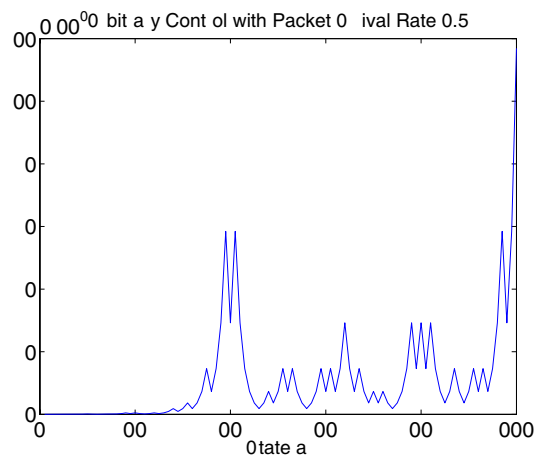


Fig. 3. Any Control with Rate Condition Not Satisfied

$R > R_{\min}$  is not satisfied, hence the closed loop system is not almost sure stable. The plot shows the system does not converge.

Figure 4 is for Lemma 8.  $M = 60$  and  $\bar{U} = 59.9999$ . The condition that  $M < \frac{1}{a-1}\bar{U}$  is not satisfied, hence the system is not stable, which is exactly captured in the plot.

The final Figure 5 is for Lemma 10. The packet arrival rate  $\gamma = 0.5$ . According to Lemma 4,  $R_{\min} = \frac{\log a}{\gamma} = 2$ . The actual network data rate  $R = 10$  which is much greater than  $R_{\min}$ . With arbitrary control inputs, the closed loop system would be almost sure stable from Lemma 4. However, we set  $M = 50$  and  $\bar{U} = 100$ . Though  $M$  and  $\bar{U}$  satisfies the relationship in Lemma 8, the closed loop system is not stable from this plot.

#### IV. CONCLUSIONS AND FUTURE WORK

In this second paper on towards a packet-based control theory, we have considered rate issues that are inherent with a networked control system. In particular, we extend the results we obtained from the first paper [16] and from

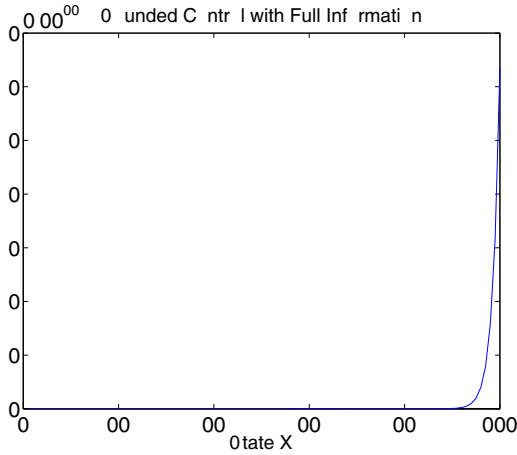


Fig. 4. Finite Control with Bound Condition Not Satisfied

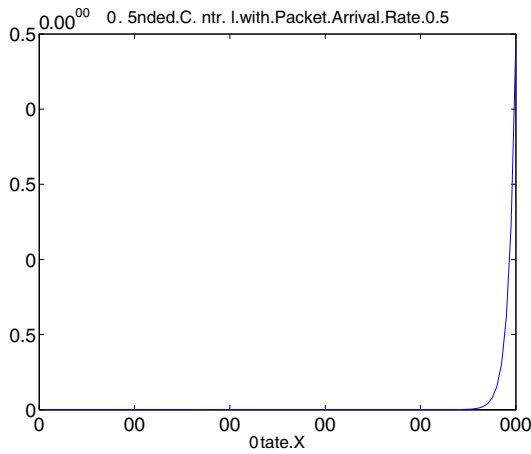


Fig. 5. Finite Control with Packet Drops

previous literatures, for example [18]. A number of rate results under different settings are given when there are packet drops, when the network has only finite data rate, and when the control inputs are upper bounded due to the physical power constraints. Simulations are provided to assist our theories.

As the next steps towards a more complete packet-based control theory, we would like to investigate the following topics. The first one is to extend all the rate results to the setting where plant dynamics involves process noises and observation noises. Noiseless systems are easy to analyze but are certainly not of practical considerations.

The second topic we would like to consider is to construct a unified framework which incorporates different issues together. Previous works on networked control systems deal with single issue at a time. For example, Nilsson in his PhD thesis [9] only considered delays induced by the network. Sinopoli and et.al in their work [2] on networked estimation only considered packet drops introduced by the network.

It would also be interesting to extend the idea in this

paper to performance issues. For example, what is the network condition which primarily includes data rate  $R$ , packet drop rate  $\gamma$ , induced delays  $D$  and so on, such that the nominal system performance is almost surely guaranteed while maintaining the almost sure stability. Those problems are of immediate interest and will be pursued in later papers centered around this topic.

#### Acknowledgment

The authors would like to thank Michael Epstein for helping simulate the closed loop system in the previous section.

#### REFERENCES

- [1] A.Abate, Ling Shi, S.Simic, and S.Sastry. A stability criterion on stochastic hybrid systems. International Symposium of Mathematical Theory of Networks and Systems, Leuven, 2004.
- [2] B.Sinopoli, L.Schenato, M.Franceschetti, K.Poola, M.Jordan, and S.Sastry. Kalman filtering with intermittent observations. *IEEE Transactions on Automatic Control*, 49(9):1453–1464, 2004.
- [3] F. Callier and C. Desoer. *Linear System Theory*. Springer Verlag, 1991.
- [4] Thomas M. Cover and Joy A. Thomas. *Elements of Information Theory*. Wiley-Interscience, 1991.
- [5] Yuguang Fang and Kenneth A. Loparo. Stochastic stability of jump linear systems. *IEEE Transactions on Automatic Control*, 47(7):1204–1208, 2002.
- [6] Robert G. Gallager. *Information Theory and Reliable Communication*. Wiley, 1968.
- [7] G.N.Nair and R.J.Evans. Communication-limited stabilization of linear systems. In *Proceedings of the 39th Conf. on Decision and Contr.*, volume 1, pages 1005–1010, Dec 2000.
- [8] I.R.Petersen and A.V.Savkin. Multi-rate stabilization of multivariable discrete-time linear systems via a limited capacity communication channel. In *Proceedings of the 40th Conf. on Decision and Contr.*, volume 1, pages 304–309, Dec 2001.
- [9] J.Nilsson. *Real Time Control Systems with Delays*. PhD thesis, Lund Institute of Technology, Lund, Sweden, 1998.
- [10] Hassan K. Khalil. *Nonlinear Systems*. Prentice Hall, second edition edition, 1995.
- [11] S. K. Mitter. Control with limited information: the role of systems theory and information theory. In *ISIT 2000 Plenary Talk, IEEE Information Theory Society Newsletter*, page 50, 2000.
- [12] R.W.Brockett and D.Liberzon. Quantized feedback stabilization of linear systems. *IEEE Trans. Automat. Contr.*, 45, July 2000.
- [13] A. Sahai. *Anytime Information Theory*. PhD thesis, Massachusetts Institute of Technology, 2001.
- [14] C. E. Shannon. A mathematical theory of communication. *Bell System Tech. J.*, 27:379–423; 623–656, 1948.
- [15] Ling Shi, Michael Epstein, Abhishek Tiwari, and Richard.M.Murray. Estimation with information loss: Asymptotic analysis and error bounds. CDC-ECC05, Accepted, 2005.
- [16] Ling Shi and Richard.M.Murray. Towards a packet-based control theory - part one: Stabilization over a packet-based network. American Control Conference, Portland, Oregon, 2005.
- [17] S.S.Sastry. *Nonlinear Systems; Analysis, Stability, and Control*. Springer Verlag, 1999.
- [18] S. C. Tatikonda. *Control Under Communication Constraints*. PhD thesis, Massachusetts Institute of Technology, 2000.
- [19] W.S.Wong and R.W.Brochett. Systems with finite communication bandwidth-part i: State estimation problems. *IEEE Trans. Automat. Contr.*, 42, Sept 1997.
- [20] W.S.Wong and R.W.Brochett. Systems with finite communication bandwidth-part ii: Stabilization with limited information feedback. *IEEE Trans. Automat. Contr.*, 44, May 1999.
- [21] Wei Zhang. *Stability Analysis of Networked Control Systems*. PhD thesis, Case Western Reserve University, 2001.